

# MORITA EQUIVALENCE METHODS IN CLASSIFICATION OF FUSION CATEGORIES

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ABSTRACT. We describe an approach to classification of fusion categories in terms of Morita equivalence. This is usually achieved by analyzing Drinfeld centers of fusion categories and finding Tannakian subcategories therein.

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## 1. INTRODUCTION

The purpose of this survey is to describe an approach to classification of fusion categories using categorical duality (i.e., categorical Morita equivalence). This duality is a categorical analogue of the following classical construction in algebra: given a ring  $R$  and a left  $R$ -module  $M$  one has the ring  $R_M^* := \text{End}_R(M)$  of  $R$ -linear endomorphisms of  $M$ . We can view this ring as the dual ring of  $R$  with respect to  $M$ . Two rings  $R$  and  $S$  are called *Morita equivalent* if  $S^{\text{op}}$  is isomorphic to  $R_M^*$  with respect to some progenerator module  $M$  (here  $S^{\text{op}}$  denotes the ring with the opposite multiplication). It is well known that Morita equivalent rings have equivalent Abelian categories of modules.

In the categorical setting one replaces rings by tensor categories, their modules by module categories, and module endomorphisms by module endofunctors, see Section 3.1 for definitions. Thus, given, a tensor category  $\mathcal{A}$  and a left  $\mathcal{A}$ -module category  $\mathcal{M}$  one has a new tensor category  $\mathcal{A}_{\mathcal{M}}^*$  which we call *categorically Morita equivalent* or *dual* to  $\mathcal{A}$  with respect to  $\mathcal{M}$ . If  $\mathcal{A}$  is a fusion category and  $\mathcal{M}$  is an indecomposable semisimple  $\mathcal{A}$ -module category then  $\mathcal{A}_{\mathcal{M}}^*$  is also a fusion category.

One can produce new examples of fusion categories in this way. Namely, starting with a known fusion category  $\mathcal{A}$  one finds its module categories and construct duals. For example, when  $\mathcal{A}$  is a pointed (respectively, nilpotent) category (see Definitions 2.7 and 2.19) the resulting dual categories form an important class of fusion categories called *group-theoretical* (respectively, *weakly group-theoretical*) categories.

In the opposite direction, given a class of fusion categories (e.g., of a given dimension) one can try to show that categories in this class are categorically Morita equivalent to some well understood categories. In other words, one tries to classify fusion categories up to a categorical Morita equivalence. Classification results of this type for fusion categories of small integral dimension were obtained in [9, 12, 30]. The techniques used in these papers are based on recovering some group-theoretical information about the Morita equivalence class of a given fusion category.

In this paper we try to explain the ideas and methods used in the above classification. Of particular importance is the structural theory of braided categories. This comes from the fact that the braided equivalence class of the Drinfeld center of a fusion category is its complete Morita equivalence invariant, see Theorem 5.1.

This paper does not contain new results or proofs. Its only goal is to collect relevant notions and facts in one place. In Sections 2, 3, and 4 we recall definitions and basic facts about fusion categories, their module categories, and braided categories. In Section 5 we describe a connection between the structure of the center of a fusion category  $\mathcal{A}$  and the Morita equivalence class of  $\mathcal{A}$ . These results are applied to classification of fusion categories of low dimension in Section 6.

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## 2. DEFINITIONS AND BASIC NOTIONS

Throughout this paper  $\mathbb{k}$  denotes an algebraically closed field of characteristic zero. All categories are assumed to be Abelian, semisimple,  $\mathbb{k}$ -linear and have finitely many isomorphism classes of simple objects and finite dimensional spaces of morphisms. All functors are assumed to be additive and  $\mathbb{k}$ -linear.

**2.1. Definitions, basic properties, and examples of fusion categories.** The following definition was given in [11].

**Definition 2.1.** A *fusion category* over  $\mathbb{k}$  is a rigid tensor category such that the unit object  $\mathbf{1}$  is simple.

That is, a fusion category  $\mathcal{A}$  is a category equipped with tensor product bifunctor  $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , the natural isomorphisms (associativity and unit constraints)

$$\begin{aligned} (1) \quad & a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \\ (2) \quad & l_X : \mathbf{1} \otimes X \xrightarrow{\sim} X, \quad \text{and} \quad r_X : X \otimes \mathbf{1} \xrightarrow{\sim} X, \end{aligned}$$

satisfying the following coherence axioms:

**1. The Pentagon Axiom.** The diagram

$$(3) \quad \begin{array}{ccc} & ((W \otimes X) \otimes Y) \otimes Z & \\ \swarrow a_{W \otimes X, Y, Z} & & \searrow a_{W, X, Y \otimes Z} \\ (W \otimes (X \otimes Y)) \otimes Z & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow a_{W, X \otimes Y, Z} & & \downarrow a_{W, X, Y \otimes Z} \\ W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\text{id}_W \otimes a_{X, Y, Z}} & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

is commutative for all objects  $W, X, Y, Z$  in  $\mathcal{A}$ .

**2. The triangle axiom.** The diagram

$$(4) \quad \begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X, \mathbf{1}, Y}} & X \otimes (\mathbf{1} \otimes Y) \\ \searrow r_X \otimes \text{id}_Y & & \swarrow \text{id}_X \otimes l_Y \\ & X \otimes Y & \end{array}$$

is commutative for all objects  $X, Y$  in  $\mathcal{A}$ .

The coherence theorem of MacLane states that *every* diagram constructed from the associativity and unit isomorphisms commutes. We will sometimes omit associativity constraints from formulas.

The rigidity condition means that for every object  $X$  of  $\mathcal{A}$  there exist left and right duals of  $X$ . Here a *left dual* of  $X$  is an object  $X^*$  in  $\mathcal{A}$  for which there exist morphisms  $\text{ev}_X : X^* \otimes X \rightarrow \mathbf{1}$  and  $\text{coev}_X : \mathbf{1} \rightarrow X \otimes X^*$ , called the *evaluation* and *coevaluation* such that the compositions

$$(5) \quad X \xrightarrow{\text{coev}_X \otimes \text{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X, X^*, X}} X \otimes (X^* \otimes X) \xrightarrow{\text{id}_X \otimes \text{ev}_X} X,$$

$$(6) \quad X^* \xrightarrow{\text{id}_{X^*} \otimes \text{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a_{X^*, X^*, X}^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{\text{ev}_X \otimes \text{id}_{X^*}} X^*$$

are the identity morphisms. A right dual  ${}^*X$  is defined in a similar way. Dual objects are unique up to isomorphism. One has  $({}^*X)^* \cong X \cong {}^*(X^*)$  for all objects  $X$  in  $\mathcal{C}$ . Also, there exist a (non-canonical) isomorphism  $X^* \cong {}^*X$  for every  $X$  (indeed, for a simple  $X$  both  $X^*$  and  ${}^*X$  are isomorphic to the unique simple object  $Y$  such that  $\mathbf{1}$  is contained in  $X \otimes Y$ ).

For a fusion category  $\mathcal{A}$  let  $\mathcal{A}^{\text{op}}$  denote the fusion category with the opposite tensor product.

**Definition 2.2.** Let  $\mathcal{A}_1, \mathcal{A}_2$  be fusion categories. A *tensor functor* between  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is a functor  $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  along with natural isomorphisms

$$J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \quad \text{and} \quad \varphi : F(\mathbf{1}) \xrightarrow{\sim} \mathbf{1}$$

such that the diagrams

$$(7) \quad \begin{array}{ccc} (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{a_{F(X), F(Y), F(Z)}} & F(X) \otimes (F(Y) \otimes F(Z)) \\ \downarrow J_{X,Y} \otimes \text{id}_{F(Z)} & & \downarrow \text{id}_{F(X)} \otimes J_{Y,Z} \\ F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\ \downarrow J_{X \otimes Y, Z} & & \downarrow J_{X, Y \otimes Z} \\ F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z)), \end{array}$$

$$(8) \quad \begin{array}{ccc} F(\mathbf{1}) \otimes F(X) & \xrightarrow{J_{\mathbf{1}, X}} & F(\mathbf{1} \otimes X) \\ \downarrow \varphi \otimes \text{id}_{F(X)} & & \downarrow F(l_X) \\ \mathbf{1} \otimes F(X) & \xrightarrow{l_{F(X)}} & F(X), \end{array}$$

and

$$(9) \quad \begin{array}{ccc} F(X) \otimes F(\mathbf{1}) & \xrightarrow{J_{X, \mathbf{1}}} & F(X \otimes \mathbf{1}) \\ \downarrow \text{id}_{F(X)} \otimes \varphi & & \downarrow F(r_X) \\ F(X) \otimes \mathbf{1} & \xrightarrow{r_{F(X)}} & F(X). \end{array}$$

commute for all objects  $X, Y, Z$  in  $\mathcal{A}_1$ .

**Definition 2.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be fusion categories and let  $F^1, F^2 : \mathcal{A} \rightarrow \mathcal{B}$  be tensor functors between fusion categories with tensor structures

$$J_{X,Y}^i : F^i(X) \otimes F^i(Y) \xrightarrow{\sim} F^i(X \otimes Y), \quad i = 1, 2.$$

A natural morphism  $\eta$  between  $F^1$  and  $F^2$  is called *tensor* if its components

$$\eta_X : F^1(X) \rightarrow F^2(X)$$

satisfy the commutative diagram

$$(10) \quad \begin{array}{ccc} F^1(X) \otimes F^1(Y) & \xrightarrow{\eta_X \otimes \eta_Y} & F^2(X) \otimes F^2(Y) \\ \downarrow J_{X,Y}^1 & & \downarrow J_{X,Y}^2 \\ F^1(X \otimes Y) & \xrightarrow{\eta_{X \otimes Y}} & F^2(X \otimes Y), \end{array}$$

for all  $X, Y \in \mathcal{A}$ .

Tensor autoequivalences of a fusion category  $\mathcal{A}$  form a monoidal category denoted  $\text{Aut}_{\otimes}(\mathcal{A})$ .

Let  $G$  be a finite group and let  $\underline{G}$  denote the monoidal category whose objects are elements of  $G$ , morphisms are identities, and the tensor product is given by the multiplication in  $G$ .

**Definition 2.4.** An *action* of  $G$  on a fusion category  $\mathcal{A}$  is a monoidal functor

$$(11) \quad T : \underline{G} \rightarrow \text{Aut}_{\otimes}(\mathcal{A}) : g \mapsto T_g.$$

This means that for every  $g \in G$  there is a tensor autoequivalence  $T_g : \mathcal{A} \rightarrow \mathcal{A}$  and for any pair  $g, h \in G$ , there is a natural isomorphism of tensor functors

$$\gamma_{g,h} : T_g \circ T_h \simeq T_{gh}$$

satisfying usual compatibility conditions.

Note that for any fusion category  $\mathcal{A}$  the functor

$$\mathcal{A} \rightarrow \mathcal{A}^{\text{op}} : X \mapsto X^*$$

is a tensor equivalence. Consequently, the functor

$$\mathcal{A} \rightarrow \mathcal{A} : X \mapsto X^{**}$$

is a tensor autoequivalence of  $\mathcal{A}$ .

**Definition 2.5.** A *pivotal structure* on a fusion category  $\mathcal{A}$  is a tensor isomorphism  $\psi$  between the identity autoequivalence of  $\mathcal{A}$  and the functor  $X \mapsto X^{**}$  of taking the second dual. A fusion category with a pivotal structure is called *pivotal*.

In a pivotal category there is a notion of a *trace* of an endomorphism. Namely, for  $f \in \text{End}_{\mathcal{A}}(X)$  set:

$$(12) \quad \text{Tr}(f) : \mathbf{1} \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{\psi_X \circ f \otimes \text{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\text{ev}_{X^*}} \mathbf{1},$$

so that  $\text{Tr}(f) \in \text{End}_{\mathcal{A}}(\mathbf{1}) = \mathbb{k}$ . The *dimension* of  $X \in \mathcal{A}$  is defined by

$$(13) \quad d_X = \text{Tr}(\text{id}_X).$$

Note that  $d_X \neq 0$  for every simple  $X$ . A pivotal structure (respectively, a pivotal category) is called *spherical* if  $d_X = d_{X^*}$  for all objects  $X$ .

**Remark 2.6.** It is not known whether every fusion category has a pivotal (or spherical) structure. It is true for pseudo-unitary categories, see Proposition 2.34. In particular it is true for categories of integer Frobenius-Perron dimension (Corollary 6.2).

We say that a tensor functor  $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  is *injective* if it is fully faithful and *surjective* if for any object  $Y \in \mathcal{A}_2$  there is an object  $X \in \mathcal{A}_1$  such that  $Y$  is isomorphic to a direct summand of  $F(X)$ . In the latter case we call  $\mathcal{A}_2$  a quotient category of  $\mathcal{A}$ .

By a *fusion subcategory* of a fusion category  $\mathcal{A}$  we always mean a full tensor subcategory  $\tilde{\mathcal{A}} \subset \mathcal{A}$  such that if  $X \in \tilde{\mathcal{A}}$  is isomorphic to a direct summand of an object of  $\tilde{\mathcal{A}}$  then  $X \in \tilde{\mathcal{A}}$ . As an additive category,  $\tilde{\mathcal{A}}$  is generated by some of the simple objects of  $\mathcal{A}$ . It is known, see [10, Appendix F], that a fusion subcategory of a fusion category is rigid; therefore, it is itself a fusion category.

Let  $\text{Vec}$  denote the fusion category of finite dimensional vector spaces over  $\mathbb{k}$ . A tensor functor  $\mathcal{A} \rightarrow \text{Vec}$  will be called a *fiber functor*.

Any fusion category  $\mathcal{A}$  contains a trivial fusion subcategory consisting of multiples of the unit object  $\mathbf{1}$ . We will identify this subcategory with  $\text{Vec}$ .

**Definition 2.7.** A fusion category is called *pointed* if all its simple objects are invertible with respect to tensor product. For a fusion category  $\mathcal{A}$  we denote  $\mathcal{A}_{pt}$  the maximal pointed fusion subcategory of  $\mathcal{A}$ .

We will denote  $\mathcal{A} \boxtimes \mathcal{B}$  the *tensor product* of fusion categories  $\mathcal{A}$  and  $\mathcal{B}$  [7, Section 5]. The category  $\mathcal{A} \boxtimes \mathcal{B}$  is obtained as the completion of the  $\mathbb{k}$ -linear direct product  $\mathcal{A} \otimes_{\mathbb{k}} \mathcal{B}$  under direct sums and subobjects.

**2.2. First examples of fusion categories.** Let  $G$  be a finite group.

**Example 2.8.** The following is the most general example of a pointed fusion category. Let  $\omega$  be a normalized 3-cocycle on  $G$  with values in  $\mathbb{k}^\times$ , the multiplicative group of the ground field. That is,  $\omega : G \times G \times G \rightarrow \mathbb{k}^\times$  is a function satisfying equations

$$(14) \quad \omega(g_1 g_2, g_3, g_4) \omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3) \omega(g_1, g_2 g_3, g_4) \omega(g_2, g_3, g_4)$$

and

$$(15) \quad \omega(g_1, 1, g_2) = 1,$$

for all  $g_1, g_2, g_3, g_4 \in G$ .

Let  $\text{Vec}_G^\omega$  denote the category of  $G$ -graded  $\mathbb{k}$ -vector spaces with the tensor product of objects  $U = \bigoplus_{g \in G} U_g$  and  $V = \bigoplus_{g \in G} V_g$  given by

$$(U \otimes V)_g = \bigoplus_{xy=g} U_x \otimes V_y, \quad g \in G,$$

with the associativity constraint  $a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$  on homogeneous spaces  $U, V, W$  of degrees  $g_1, g_2, g_3 \in G$  given by  $\omega(g_1, g_2, g_3)$  times the canonical vector spaces isomorphism  $(U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$ .

For any  $g \in G$  let  $\delta_g$  denote the corresponding simple object of  $\text{Vec}_G^\omega$ . We have

$$\delta_g \otimes \delta_h = \delta_{gh}, \quad g, h \in G.$$

Two categories  $\text{Vec}_G^\omega$  and  $\text{Vec}_{\tilde{G}}^{\tilde{\omega}}$  are equivalent if and only if there is a group isomorphism  $f : G \rightarrow \tilde{G}$  such that  $\omega$  and  $\tilde{\omega} \circ (f \times f \times f)$  are cohomologous 3-cocycles on  $G$ .

When  $\omega = 1$  we will denote the corresponding pointed fusion category by  $\text{Vec}_G$ .

**Example 2.9.** Let  $\text{Rep}(G)$  be the category of finite dimensional representations of  $G$  over  $\mathbb{k}$ . It is a fusion category with the usual tensor product and associativity isomorphisms. The unit object is the trivial representation. The left and right dual objects of the representation  $V$  are both given by the dual representation  $V^*$ . Simple objects of  $\text{Rep}(G)$  are irreducible representations.

The category  $\text{Rep}(G)$  is pointed if and only if  $G$  is Abelian, in which case there is a canonical equivalence  $\text{Rep}(G) \cong \text{Vec}_{\hat{G}}$ , where  $\hat{G}$  is the group of characters of  $G$ .

**Example 2.10.** This is a generalization of Example 2.9. Let  $H$  be a semisimple Hopf algebra over  $\mathbb{k}$  (such an algebra is automatically finite dimensional) with the comultiplication  $\Delta : H \rightarrow H \otimes H$ , antipode  $S : H \rightarrow H$ , and counit  $\varepsilon : H \rightarrow \mathbb{k}$  [25]. The category  $\text{Rep}(H)$  of finite dimensional  $H$ -modules is a fusion category with the tensor product of  $H$ -modules  $V, W$  being  $V \otimes_{\mathbb{k}} W$  with the action of  $H$  given by

$$h \cdot (v \otimes w) = \Delta(h)(v \otimes w), \quad v \in V, w \in W, h \in H.$$

The unit object is  $\mathbb{k}$  with the action given by  $\varepsilon$ . The left dual of an  $H$ -module  $V$  is the dual vector space  $V^*$  with an  $H$ -module structure given by  $\langle h \cdot \phi, v \rangle = \langle \phi, S(h) \cdot v \rangle$ .

Let  $H_1, H_2$  be semisimple Hopf algebras. A homomorphism  $f : H_1 \rightarrow H_2$  of Hopf algebras induces a tensor functor  $F : \text{Rep}(H_2) \rightarrow \text{Rep}(H_1)$ . The functor  $F$  is injective (respectively, surjective) if and only if  $f$  is surjective (respectively, injective).

**Remark 2.11.** Note that there is forgetful tensor functor  $\text{Rep}(H) \rightarrow \text{Vec}$ , i.e., a *fiber functor*. Conversely, if  $\mathcal{A}$  is a fusion category that admits a fiber functor then  $\mathcal{A} \cong \text{Rep}(H)$  for some semisimple Hopf algebra  $H$  [41].

**Example 2.12.** Similarly, if  $Q$  is a semisimple quasi-Hopf algebra then  $\text{Rep}(Q)$  is a fusion category. In this case the associativity constraint is determined by the associator  $\Phi \in Q \otimes Q \otimes Q$ . The category  $\text{Vec}_G^\omega$  from Example 2.8 is a special case of such category.

**Example 2.13.** Let  $\mathfrak{g}$  be a finite dimensional simple Lie algebra and let  $\hat{\mathfrak{g}}$  be the corresponding affine Lie algebra. For any  $k \in \mathbb{Z}_{>0}$  let  $\mathcal{C}(\mathfrak{g}, k)$  the category of highest weight integrable  $\hat{\mathfrak{g}}$ -modules of level  $k$  is a fusion category, see e.g. [3, Section 7.1] where this category is denoted  $\mathcal{O}_k^{\text{int}}$ .

**2.3. Group theoretical constructions: extensions and equivariantizations.** For a fusion category  $\mathcal{A}$  let  $\mathcal{O}(\mathcal{A})$  denote the set of (representatives of isomorphism classes of) simple objects of  $\mathcal{A}$ .

**Definition 2.14.** A *grading* of  $\mathcal{A}$  by a group  $G$  is a map  $\deg : \mathcal{O}(\mathcal{A}) \rightarrow G$  with the following property: for all simple objects  $X, Y, Z \in \mathcal{A}$  such that  $X \otimes Y$  contains  $Z$  one has  $\deg Z = \deg X \cdot \deg Y$ .

The name “grading” is also used for the corresponding decomposition

$$(16) \quad \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

where  $\mathcal{A}_g$  is the full additive subcategory generated by simple objects of degree  $g \in G$ . We say that a grading is *faithful* if  $\mathcal{A}_g \neq 0$  for all  $g \in G$ . Note that the *trivial component*  $\mathcal{A}_e$  of the grading (16) is a fusion subcategory of  $\mathcal{A}$ .

**Definition 2.15.** Let  $\mathcal{A}$  be a fusion category. The *adjoint* subcategory  $\mathcal{A}_{ad} \subset \mathcal{A}$  is the fusion subcategory of  $\mathcal{A}$  generated by objects  $X \otimes X^*$ ,  $X \in \mathcal{O}(\mathcal{A})$ .

It was explained in [18] that there exists a group  $U(\mathcal{A})$ , called the *universal grading group* of  $\mathcal{A}$ , and a faithful grading

$$\mathcal{A} = \bigoplus_{g \in U(\mathcal{A})} \mathcal{A}_g \quad \text{with } \mathcal{A}_e = \mathcal{A}_{ad}.$$

This grading is universal in the sense that any faithful grading (16) of  $\mathcal{A}$  is obtained by taking a quotient of the group  $U(\mathcal{A})$ .

**Example 2.16.** Let  $\mathcal{A} = \text{Rep}(H)$ , where  $H$  is a semisimple Hopf algebra. Let  $K$  be the maximal Hopf subalgebra of  $H$  contained in the center of  $H$ . Then  $K$  is isomorphic to the Hopf algebra of functions on  $U(\text{Rep}(H))$ . In other words, the universal grading group of  $\text{Rep}(H)$  is the spectrum of  $K$  [18].

**Definition 2.17.** Let  $G$  be a finite group. We say that a fusion category  $\mathcal{A}$  is a *G-extension* of a fusion category  $\mathcal{B}$  if there is a faithful  $G$ -grading of  $\mathcal{A}$  such that  $\mathcal{A}_e \cong \mathcal{B}$ .

**Remark 2.18.** A classification of  $G$ -extensions of fusion categories is obtained in [13]. Namely,  $G$ -extensions of a fusion category  $\mathcal{A}$  correspond to homomorphisms  $G \rightarrow \text{BrPic}(\mathcal{A})$ , where  $\text{BrPic}(\mathcal{A})$  is the *Brauer-Picard* group of  $\mathcal{A}$  consisting of invertible  $\mathcal{A}$ -module categories, and certain cohomological data. See [13] for details.

This theory extends the classical theory of strongly graded rings (also known as generalized crossed products) and their description using bimodules and crossed systems.

**Definition 2.19.** A fusion category  $\mathcal{A}$  is called *nilpotent* [18] if there is a sequence of finite groups  $G_1, \dots, G_n$  and a sequence of fusion subcategories of  $\mathcal{A}$ :

$$\mathcal{A}_0 = \mathbf{Vec} \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_n = \mathcal{A},$$

such that  $\mathcal{A}_i$  is a  $G_i$ -extension of  $\mathcal{A}_{i-1}$ ,  $i = 1, \dots, n$ . The smallest such  $n$  is called the *nilpotency class* of  $\mathcal{A}$ . A nilpotent fusion category is called *cyclically nilpotent* if all groups  $G_i$  are cyclic.

**Remark 2.20.** The category  $\mathbf{Vec}_G^\omega$  is nilpotent for any finite group  $G$  (since pointed fusion categories are precisely nilpotent fusion categories of class 1). The category  $\mathbf{Rep}(G)$  is nilpotent if and only if  $G$  is nilpotent.

Let  $G$  be a group acting on a fusion category  $\mathcal{A}$ , see Definition 2.4.

**Definition 2.21.** A  $G$ -equivariant object in  $\mathcal{A}$  is a pair  $(X, \{u_g\}_{g \in G})$  consisting of an object  $X$  of  $\mathcal{A}$  together with a collection of isomorphisms  $u_g : T_g(X) \simeq X$ ,  $g \in G$ , such that the diagram

$$\begin{array}{ccc} T_g(T_h(X)) & \xrightarrow{T_g(u_h)} & T_g(X) \\ \gamma_{g,h}(X) \downarrow & & \downarrow u_g \\ T_{gh}(X) & \xrightarrow{u_{gh}} & X \end{array}$$

commutes for all  $g, h \in G$ . One defines morphisms of equivariant objects to be morphisms in  $\mathcal{A}$  commuting with  $u_g$ ,  $g \in G$ .

Equivariant objects in  $\mathcal{A}$  form a fusion category, called the *equivariantization* of  $\mathcal{A}$  and denoted by  $\mathcal{A}^G$ . There is a natural forgetful tensor functor  $\mathcal{A}^G \rightarrow \mathcal{A}$ .

**Example 2.22.** Let  $G$  be a finite group.

- (i) Consider  $\mathbf{Vec}$  with the trivial action of  $G$ . Then  $\mathbf{Vec}^G \cong \mathbf{Rep}(G)$ .
- (ii) More generally, let  $N$  be a normal subgroup of  $G$ . The corresponding action of  $G/N$  on  $N$  induces an action of  $G/N$  on  $\mathbf{Rep}(N)$ . We have  $\mathbf{Rep}(N)^{G/N} \cong \mathbf{Rep}(G)$ .
- (iii) Consider  $\mathbf{Vec}_G$  with the action of  $G$  by conjugation. Then  $(\mathbf{Vec}_G)^G \cong \mathcal{Z}(\mathbf{Vec}_G)$  is the center of  $\mathbf{Vec}_G$ , cf. Example 4.10.

**2.4. The Grothendieck ring and Frobenius-Perron dimensions.** As before, let  $\mathcal{O}(\mathcal{A})$  denote the set of isomorphism classes of simple objects in a fusion category  $\mathcal{A}$ .

For any object  $X$  of  $\mathcal{A}$  and any  $Y \in \mathcal{O}(\mathcal{A})$  let  $[X : Y]$  denote the multiplicity of  $Y$  in  $X$ .

The *Grothendieck ring*  $K(\mathcal{A})$  of a fusion category  $\mathcal{A}$  is generated by isomorphism classes of objects  $X \in \mathcal{A}$  with the addition and multiplication given by

$$X + Y = X \oplus Y \quad \text{and} \quad XY = X \otimes Y$$

for all  $X, Y \in \mathcal{A}$ . Clearly,  $K(\mathcal{A})$  is a free  $\mathbb{Z}$ -module with basis  $\mathcal{O}(\mathcal{A})$ .

**Example 2.23.** For the pointed fusion category  $\mathbf{Vec}_G^\omega$  from Example 2.8 we have  $K(\mathbf{Vec}_G^\omega) = \mathbb{Z}G$ .



There exists a unique ring homomorphism  $\text{FPdim} : K(\mathcal{A}) \rightarrow \mathbb{R}$ , called the *Frobenius-Perron dimension* such that  $\text{FPdim}(X) > 0$  for any  $0 \neq X \in \mathcal{A}$ , see [11, Section 8.1]. The number  $\text{FPdim}(X)$  is the largest positive eigenvalue (i.e., the Frobenius-Perron eigenvalue) of the integer non-negative matrix

$$N^X = (N_{XY}^Z)_{Y, Z \in \mathcal{O}(\mathcal{A})},$$

where

$$X \otimes Y \cong \bigoplus_{Z \in \mathcal{O}(\mathcal{A})} N_{XY}^Z Z.$$

For every object  $X \in \mathcal{A}$  we have

$$\text{FPdim}(X) = \text{FPdim}(X^*).$$

For a fusion category  $\mathcal{A}$  one defines (see [11, Section 8.2]) its *Frobenius-Perron dimension*:

$$(17) \quad \text{FPdim}(\mathcal{A}) = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X)^2.$$

We define the *virtual regular object* of  $\mathcal{A}$  as

$$R_{\mathcal{A}} := \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X) X \in K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Note that  $R_{\mathcal{A}}$  is the unique (up to a non-zero scalar multiple) element of  $K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$  such that  $XR_{\mathcal{A}} = R_{\mathcal{A}}X = \text{FPdim}(X)R_{\mathcal{A}}$  for all  $X \in K(\mathcal{A})$ .

Clearly,  $\text{FPdim}(\mathcal{A})$  and  $\text{FPdim}(X)$  for non-zero  $X \in \mathcal{A}$  are positive algebraic integers. It was shown in [11, Corollary 8.54] that they are, in fact, cyclotomic integers.

**Definition 2.24.** We say that a fusion category  $\mathcal{A}$  is *integral* if  $\text{FPdim}(X) \in \mathbb{Z}$  for every object  $X \in \mathcal{A}$ .

**Remark 2.25.** The Frobenius-Perron dimensions in categories considered in Examples 2.8 – 2.12 coincide with vector space dimensions, so these categories are integral. Conversely, if  $\mathcal{A}$  is an integral fusion category then  $\mathcal{A}$  is equivalent to the representation category of some semisimple quasi-Hopf algebra  $Q$  (note that this  $Q$  is not unique) by [11, Theorem 8.33]. In this case we can view  $R_{\mathcal{A}}$  as an element of  $K(\mathcal{A})$ , namely as the class of the regular representation of  $Q$ .

**Remark 2.26.** Note a difference between Frobenius-Perron dimensions and dimensions defined by formula (12). The former takes values in  $\mathbb{R}$  while the latter take values in  $\mathbb{k}$ . So these dimensions are not equal in general.

**Theorem 2.27.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a surjective tensor functor between fusion categories. Then the ratio  $\text{FPdim}(\mathcal{A})/\text{FPdim}(\mathcal{B})$  is an algebraic integer  $\geq 1$ .

*Proof.* Note that  $F$  induces a surjective algebra homomorphism

$$f : K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow K(\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

We have  $f(R_{\mathcal{A}}) = aR_{\mathcal{B}}$  for some non-zero  $a \in \mathbb{R}$ . Computing the multiplicity of  $\mathbf{1}$  in both sides of the last equality we have

$$a = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X)[F(X) : \mathbf{1}].$$

On the other hand, since  $f$  preserves Frobenius-Perron dimensions, we have

$$a = \frac{\text{FPdim}(R_{\mathcal{A}})}{\text{FPdim}(R_{\mathcal{B}})} = \frac{\text{FPdim}(\mathcal{A})}{\text{FPdim}(\mathcal{B})}.$$

Comparing last two equalities gives the result.  $\square$

**Remark 2.28.** In Theorem 2.27 one has  $\text{FPdim}(\mathcal{A}) = \text{FPdim}(\mathcal{B})$  if and only if  $F$  is an equivalence.

A consequence of Theorem 2.27 is the following formula for the Frobenius-Perron dimension of the equivariantization category:

$$\text{FPdim}(\mathcal{A}^G) = |G| \text{FPdim}(\mathcal{A}).$$

The following result is an analogue of Lagrange's Theorem in theory of groups and Hopf algebras, see [11, Proposition 8.15] and [15, Theorem 3.47].

**Theorem 2.29.** *Let  $\mathcal{A}$  be a fusion category and let  $\mathcal{B} \subset \mathcal{A}$  be a fusion subcategory. Then the ratio  $\text{FPdim}(\mathcal{B})/\text{FPdim}(\mathcal{A})$  is an algebraic integer  $\leq 1$ .*

In particular, if  $\mathcal{A}$  is a faithful  $G$ -extension of  $\mathcal{B}$  then

$$\text{FPdim}(\mathcal{A}) = |G| \text{FPdim}(\mathcal{B}).$$

**2.5. Ocneanu's rigidity.** The statement that a fusion category cannot be deformed is known as the Ocneanu rigidity because its formulation and proof for unitary categories was suggested (but not published) by Ocneanu. The following result was proved in [11, Theorems 2.28 and 2.31].

**Theorem 2.30.** *The number of fusion categories with a given Grothendieck ring is finite. The number of (equivalence classes of) tensor functors between a fixed pair of fusion categories is finite.*

**Remark 2.31.** It follows from Theorem 2.30 that every fusion category  $\mathcal{A}$  is defined over an algebraic number field. That is, the structure constants (6j symbols) of  $\mathcal{A}$  can be written using algebraic numbers. Therefore, for any  $g \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  there is a category  $g(\mathcal{A})$ , the *Galois conjugate* of  $\mathcal{A}$  obtained from  $\mathcal{A}$  by conjugating its structure constants (6j symbols) by  $g$ .

**Corollary 2.32.** *For every positive number  $M$  the number of fusion categories whose Frobenius-Perron dimension is  $\leq M$  is finite.*

*Proof.* In view of Theorem 2.30 it suffices to show that the number of possible Grothendieck rings of fusion categories of Frobenius-Perron dimension  $\leq M$  is finite. For any simple objects  $X, Y, Z$  let  $N_{XY}^Z$  denote the multiplicity of  $Z$  in  $X \otimes Y$ . In any fusion category of Frobenius-Perron dimension  $\leq M$  we have

$$N_{XY}^Z \leq \frac{\text{FPdim}(X) \text{FPdim}(Y)}{\text{FPdim}(Z)} \leq \frac{M \text{FPdim}(Y)}{\text{FPdim}(Z)},$$

whence  $(N_{XY}^Z)^2 = N_{XY}^Z N_{X^*Y}^Y \leq M^2$ . Thus the structure constants of the Grothendieck rings of the above class of fusion categories are uniformly bounded by  $M$ , so there are finitely many Grothendieck rings.  $\square$

**2.6. Categorical dimension and pseudo-unitary categories.** Let  $\mathcal{A}$  be a fusion category and let  $X$  be an object of  $\mathcal{A}$ . Given a morphism  $a_X : X \rightarrow X^{**}$  we define its trace  $\text{Tr}(a_X)$  similarly to how it was done in (12):

$$\text{Tr}(a_X) : \mathbf{1} \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{a_X \otimes \text{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\text{ev}_X} \mathbf{1}.$$

If  $X$  is simple and  $a_X : X \rightarrow X^{**}$  is an isomorphism then  $\text{Tr}(a_X) \neq 0$  and the quantity  $|X|^2 = \text{Tr}(a_X) \text{Tr}((a_X^{-1})^*) \neq 0$  does not depend on the choice of  $a_X$  (here  $|X|^2$  is regarded as a single symbol and not as the square of a modulus).

Define the *categorical dimension* of  $\mathcal{A}$  by

$$(18) \quad \dim(\mathcal{A}) = \sum_{X \in \mathcal{O}(\mathcal{A})} |X|^2.$$

Suppose  $\mathbb{k} = \mathbb{C}$ , the field of complex numbers. Then one can show that  $|X|^2 > 0$  for every simple object  $X$  in  $\mathcal{A}$  and, consequently,

$$(19) \quad \dim(\mathcal{A}) \neq 0$$

for any fusion category  $\mathcal{A}$ , see [11, Theorem 2.3]. Furthermore, for every simple  $X \in \mathcal{A}$  one has

$$(20) \quad |X|^2 \leq \text{FPdim}(X)^2$$

(see [11, Proposition 8.21]) and, hence, the categorical dimension of  $\mathcal{A}$  is dominated by its Frobenius-Perron dimension :

$$(21) \quad \dim(\mathcal{A}) \leq \text{FPdim}(\mathcal{A}).$$

**Definition 2.33.** A fusion category  $\mathcal{A}$  over  $\mathbb{C}$  is called *pseudo-unitary* if its categorical and Frobenius-Perron dimensions are equal, i.e.,  $\dim(\mathcal{A}) = \text{FPdim}(\mathcal{A})$ .

It follows from (20) that if  $\mathcal{A}$  is pseudo-unitary then  $|X|^2 = \text{FPdim}(X)^2$  for every simple object  $X$ .

It is known [14] that in any fusion category there is a canonical natural tensor isomorphism  $g_X : X \xrightarrow{\sim} X^{****}$ . Let  $a_X : X \xrightarrow{\sim} X^{**}$  be a natural isomorphism such that  $a_{X^{**}} \circ a_X = g_X$  (i.e.,  $a_X$  is a square root of  $g_X$ ). For all  $X, Y, V \in \mathcal{O}(\mathcal{A})$  let

$$(22) \quad b_{XY}^V : \text{Hom}_{\mathcal{A}}(V, X \otimes Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(V^{**}, X^{**} \otimes Y^{**})$$

be a linear isomorphism such that

$$a_X \otimes a_Y = \bigoplus_{V \in \mathcal{O}(\mathcal{A})} b_{XY}^V \otimes a_V.$$

Note that the source and target of (22) are canonically isomorphic so that we can regard  $b_{XY}^V$  as an automorphism of  $\text{Hom}_{\mathcal{A}}(V, X \otimes Y)$ . The natural isomorphism  $a_X$  is tensor (i.e., is a pivotal structure) if and only if  $b_{XY}^V = \text{id}$  for all  $X, Y, V \in \mathcal{O}(\mathcal{A})$ . Since  $a_X$  is a square root of a tensor isomorphism  $g_X$  we see that  $(b_{XY}^V)^2 = \text{id}$ .

The integers

$$N_{XY}^V = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{A}}(V, X \otimes Y) \quad \text{and} \quad T_{XY}^V = \text{Trace}(b_{XY}^V),$$

where  $\text{Trace}$  denotes the trace of a linear transformation, satisfy inequality

$$(23) \quad |T_{XY}^V| \leq N_{XY}^V.$$

The equality  $T_{XY}^V = N_{XY}^V$  occurs if and only  $b_{XY}^V = \text{id}$ , i.e., if and only if  $a_X$  is a pivotal structure.

For any  $X \in \mathcal{O}(\mathcal{A})$  let  $d_X = \text{Tr}(a_X)$  then

$$d_X d_Y = \sum_{V \in \mathcal{O}(\mathcal{A})} T_{XY}^V d_V.$$

Furthermore,  $|X|^2 = |d_X|^2$  for every  $X \in \mathcal{O}(\mathcal{A})$ .

**Proposition 2.34.** *A pseudo-unitary fusion category admits a unique spherical structure  $a_X : X \xrightarrow{\sim} X^{**}$  with respect to which  $d_X = \text{FPdim}(X)$  for every simple object  $X$ .*

*Proof.* Let  $\mathcal{A}$  be a pseudo-unitary fusion category. Let  $g_X : X \xrightarrow{\sim} X^{****}$  be a tensor isomorphism and  $a_X : X \xrightarrow{\sim} X^{**}$  be its square root as above. The idea of the proof is to twist  $g_X$  by an appropriate tensor automorphism of the identity endofunctor of  $\mathcal{A}$  in such a way that the dimensions corresponding to the square root of the resulting isomorphism become positive real numbers.

We have  $|d_X| = \text{FPdim}(X)$  for any simple object  $X$ , therefore, using (23) we obtain:

$$\begin{aligned} \text{FPdim}(X) \text{FPdim}(Y) &= |d_X d_Y| = \left| \sum_{V \in \mathcal{O}(\mathcal{A})} T_{XY}^V d_V \right| \\ &\leq \sum_{V \in \mathcal{O}(\mathcal{A})} N_{XY}^V \text{FPdim}(V) = \text{FPdim}(X) \text{FPdim}(Y), \end{aligned}$$

for all  $X, Y \in \mathcal{O}(\mathcal{A})$ . Hence, the inequality in the above chain is an equality, i.e.,  $T_{XY}^V = \pm N_{XY}^V$  and the ratio  $\frac{d_X d_Y}{d_V}$  is a real number whenever  $N_{XY}^V \neq 0$ . Thus,  $\frac{d_X^2 d_Y^2}{d_V^2}$  is a positive number whenever  $V$  is contained in  $X \otimes Y$ .

The latter property is equivalent to  $\sigma_X := \frac{|d_X|^2}{d_X^2} \text{id}_X$  being a tensor automorphism of the identity endofunctor of  $\mathcal{A}$ . Let us replace  $g_X$  by  $g_X \circ \sigma_X$  (so it is still a tensor isomorphism  $X \xrightarrow{\sim} X^{****}$ ). The square root of the latter is  $a_X \circ \tau_X$ , where  $\tau_X = \frac{|d_X|}{d_X} \text{id}_X$ . The dimensions corresponding to  $a_X \circ \tau_X$  are now such that  $d_X = |d_X|$ , i.e., are positive real numbers. This forces  $T_{XY}^V = N_{XY}^V$ . Thus,  $a_X \circ \tau_X$  is a spherical structure on  $\mathcal{A}$ .  $\square$

### 3. MODULE CATEGORIES AND CATEGORICAL MORITA EQUIVALENCE

**3.1. Definitions and examples.** Let  $\mathcal{A}$  be a fusion category.

**Definition 3.1.** A *left module category* over  $\mathcal{A}$  is a category  $\mathcal{M}$  equipped with an *action (or module product) bifunctor*  $\otimes : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$  along natural isomorphisms

$$(24) \quad m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M),$$

and

$$(25) \quad u_M : \mathbf{1} \otimes M \xrightarrow{\sim} M,$$

called *module associativity* and *unit constraints* such that the the following diagrams:

$$(26) \quad \begin{array}{ccc} & ((X \otimes Y) \otimes Z) \otimes M & \\ a_{X,Y,Z} \otimes \text{id}_M \swarrow & & \searrow m_{X \otimes Y, Z, M} \\ (X \otimes (Y \otimes Z)) \otimes M & & (X \otimes Y) \otimes (Z \otimes M) \\ m_{X, Y \otimes Z, M} \downarrow & & \downarrow m_{X, Y, Z \otimes M} \\ X \otimes ((Y \otimes Z) \otimes M) & \xrightarrow{\text{id}_X \otimes m_{Y, Z, M}} & X \otimes (Y \otimes (Z \otimes M)) \end{array}$$

and

$$(27) \quad \begin{array}{ccc} (X \otimes \mathbf{1}) \otimes M & \xrightarrow{m_{X, \mathbf{1}, M}} & X \otimes (\mathbf{1} \otimes M) \\ & \searrow r_X \otimes \text{id}_M & \swarrow \text{id}_X \otimes u_M \\ & X \otimes M & \end{array}$$

commute for all objects  $X, Y, Z$  in  $\mathcal{A}$  and  $M$  in  $\mathcal{M}$ .

**Definition 3.2.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be two module categories over  $\mathcal{A}$  with module associativity constraints  $m$  and  $n$ , respectively. An  $\mathcal{A}$ -module functor from  $\mathcal{M}$  to  $\mathcal{N}$  is a pair  $(F, s)$ , where  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a functor and

$$s_{X,M} : F(X \otimes M) \rightarrow X \otimes F(M),$$

is a natural isomorphism such that the following diagrams

$$(28) \quad \begin{array}{ccc} & F((X \otimes Y) \otimes M) & \\ F(m_{X,Y,M}) \swarrow & & \searrow s_{X \otimes Y, M} \\ F(X \otimes (Y \otimes M)) & & (X \otimes Y) \otimes F(M) \\ s_{X, Y \otimes M} \downarrow & & \downarrow n_{X, Y, F(M)} \\ X \otimes F(Y \otimes M) & \xrightarrow{\text{id}_X \otimes s_{Y, M}} & X \otimes (Y \otimes F(M)) \end{array}$$

and

$$(29) \quad \begin{array}{ccc} F(\mathbf{1} \otimes M) & \xrightarrow{s_{\mathbf{1}, M}} & \mathbf{1} \otimes F(M) \\ & \searrow F(l_M) & \swarrow l_{F(M)} \\ & F(M) & \end{array}$$

commute for all  $X, Y \in \mathcal{C}$  and  $M \in \mathcal{M}$ .

A *module equivalence* of  $\mathcal{A}$ -module categories is an  $\mathcal{A}$ -module functor that is an equivalence of categories.

**Definition 3.3.** A morphism between  $\mathcal{A}$ -module functors  $(F, s)$  and  $(G, t)$  is a natural transformation  $\nu$  from  $F$  to  $G$  such that the following diagram commutes

for any  $X \in \mathcal{A}$  and  $M \in \mathcal{M}$ :

$$(30) \quad \begin{array}{ccc} F(X \otimes M) & \xrightarrow{s_{X,M}} & X \otimes F(M) \\ \nu_{X \otimes M} \downarrow & & \downarrow \text{id}_X \otimes \nu_M \\ G(X \otimes M) & \xrightarrow{t_{X,M}} & X \otimes G(M). \end{array}$$

An  $\mathcal{A}$ -module category is called *indecomposable* if it is not equivalent to a direct sum of two non-trivial  $\mathcal{A}$ -module categories.

Every  $\mathcal{A}$ -module category is completely reducible, i.e., if  $\mathcal{M}$  is an  $\mathcal{A}$ -module category and  $\mathcal{N} \subset \mathcal{M}$  is a full  $\mathcal{A}$ -module subcategory then there exists a full  $\mathcal{A}$ -module subcategory  $\mathcal{N}' \subset \mathcal{M}$  such that  $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}'$ .

A typical example of a left  $\mathcal{A}$ -module category is the category  $\mathcal{A}_A$  of right modules over a separable algebra  $A$  in  $\mathcal{A}$  [34].

**Example 3.4.** Let  $G$  be a finite group and let  $L \subset G$  be a subgroup, and let  $\psi \in Z^2(L, \mathbb{k}^\times)$  be a 2-cocycle on  $L$ . By definition, a *projective representation* of  $L$  on a vector space  $V$  with the *Schur multiplier*  $\psi$  is a map  $\rho : G \rightarrow GL(V)$  such that  $\rho(gh) = \psi(g, h)\rho(g)\rho(h)$  for all  $g, h \in L$ . Let  $\text{Rep}_\psi(L)$  denote the abelian category of projective representations of  $L$  with the Schur multiplier  $\psi$ . The usual tensor product and associativity and unit constraints determine on  $\text{Rep}_\psi(L)$  the structure of a  $\text{Rep}(G)$ -module category. It is known that any indecomposable  $\text{Rep}(G)$ -module category is equivalent to one of this form [34].

**Example 3.5.** Let  $\mathcal{C} = \text{Vec}_G$ , where  $G$  is a group. In this case, a module category  $\mathcal{M}$  over  $\mathcal{C}$  is an abelian category  $\mathcal{M}$  with a collection of functors

$$F_g : \mathcal{M} \rightarrow \mathcal{M} \quad \delta_g \otimes M : \mathcal{M} \rightarrow \mathcal{M},$$

along with a collection of tensor functor isomorphisms

$$\eta_{g,h} : F_g \circ F_h \rightarrow F_{gh}, \quad g, h \in G,$$

satisfying the 2-cocycle relation:  $\eta_{gh,k} \circ \eta_{gh} = \eta_{g,hk} \circ \eta_{hk}$  as natural isomorphisms  $F_g \circ F_h \circ F_k \xrightarrow{\sim} F_{ghk}$  for all  $g, h, k \in G$ .

Thus, a module category over  $\text{Vec}_G$  is the same thing as an abelian category with an action of  $G$ , cf. Definition 2.4.

Let us describe indecomposable  $\text{Vec}_G$ -module categories explicitly. In any such category  $\mathcal{M}$  the set of simple objects is a transitive  $G$ -set  $X = G/L$ , where a subgroup  $L \subset G$  is determined up to a conjugacy. Let us identify the space of functions  $\text{Fun}(G/L, \mathbb{k}^\times)$  with the coinduced module  $\text{Coind}_L^G \mathbb{k}^\times$ . The  $\text{Vec}_G$ -module associativity constraint on  $\mathcal{M}$  defines a function

$$\Psi : G \times G \rightarrow \text{Coind}_L^G \mathbb{k}^\times.$$

The pentagon axiom (26) says that  $\Psi \in Z^2(G, \text{Coind}_L^G \mathbb{k}^\times)$ . Clearly, the equivalence class of  $\mathcal{M}$  depends only on the cohomology class of  $\Psi$  in  $H^2(G, \text{Coind}_L^G \mathbb{k}^\times)$ . By Shapiro's Lemma the restriction map

$$Z^2(G, \text{Coind}_L^G \mathbb{k}^\times) \rightarrow Z^2(L, \mathbb{k}^\times) : \Psi \mapsto \psi$$

induces an isomorphism  $H^2(G, \text{Coind}_L^G \mathbb{k}^\times) \xrightarrow{\sim} H^2(L, \mathbb{k}^\times)$ .

Thus, an indecomposable  $\text{Vec}_G$ -module category is determined by a pair  $(L, \psi)$ , where  $L \subset G$  is a subgroup and  $\psi \in H^2(L, \mathbb{k}^\times)$ . Let  $\mathcal{M}(L, \psi)$  denote the corresponding category.

**Remark 3.6.** Note that indecomposable  $\text{Rep}(G)$ -module categories in Example 3.4 and indecomposable  $\text{Vec}_G$ -module categories in Example 3.5 are parameterized by the same data. We will see in Section 3.2 that this is not merely a coincidence.

**Example 3.7.** This is a generalization of Example 3.5. Here we describe indecomposable module categories over pointed fusion categories. Recall that the latter categories are equivalent to  $\text{Vec}_G^\omega$  for some finite group  $G$  and a 3-cocycle  $\omega \in Z^3(G, \mathbb{k}^\times)$ .

Equivalence classes of indecomposable right  $\text{Vec}_G^\omega$ -module categories correspond to pairs  $(L, \psi)$ , where  $L$  is a subgroup of  $G$  such that  $\omega|_{L \times L \times L}$  is cohomologically trivial and  $\psi \in C^2(L, \mathbb{k}^\times)$  is a 2-cochain satisfying  $\delta^2 \psi = \omega|_{L \times L \times L}$ . The corresponding  $\text{Vec}_G^\omega$ -module category is constructed as follows. Given a pair  $(L, \psi)$  as above define an algebra

$$(31) \quad A(L, \psi) = \bigoplus_{a \in L} \delta_a$$

in  $\text{Vec}_G^\omega$  with the multiplication

$$(32) \quad \bigoplus_{a, b \in L} \psi(a, b) \text{id}_{\delta_{ab}} : A(L, \psi) \otimes A(L, \psi) \rightarrow A(L, \psi).$$

Let  $\mathcal{M}(L, \psi)$  denote the category of left  $A(L, \psi)$ -modules in  $\text{Vec}_G^\omega$ . Any  $\text{Vec}_G^\omega$ -module category is equivalent to some  $\mathcal{M}(L, \psi)$ .

**Remark 3.8.** Two  $\text{Vec}_G^\omega$ -module categories  $\mathcal{M}(L, \psi)$  and  $\mathcal{M}(L', \psi')$  are equivalent if and only if there is  $g \in G$  such that  $L' = gLg^{-1}$  and  $\psi'$  is cohomologous to  $\psi^g$  in  $H^2(L', \mathbb{k}^\times)$ , where  $\psi^g(x, y) := \psi(gxg^{-1}, gyg^{-1})$  for all  $x, y \in L$ . Here we abuse notation and identify  $\psi$  and  $\psi'$  with cocycles representing them.

**Example 3.9.** Let  $H$  be a semisimple Hopf algebra. Example 3.5 was generalized in [2] where it was shown that indecomposable  $\text{Rep}(H)$ -module categories are classified by left  $H$ -comodule algebras that are  $H$ -simple from the right and with the trivial space of coinvariants. This is another generalization of Example 3.5.

Here is another generalization of Example 3.5.

Let  $H$  be a semisimple Hopf algebra and let  $B \subset A$  be a left faithfully flat  $H$ -Galois extension with  $B$  semisimple. Let  $\mathcal{M}_B$  and  $\mathcal{M}^H$  denote the fusion categories of right  $B$ -modules and right  $H$ -comodules, respectively. Recall that the category of right Hopf  $(H, A)$ -modules  $\mathcal{M}_A^H$  is by definition the category of right  $A$ -modules over  $\mathcal{M}^H$ . By Schneider's structure theorem [38] the functor

$$\mathcal{M}_B \rightarrow (\mathcal{M}^H)_A : M \mapsto M \otimes_B A,$$

is a category equivalence with inverse  $M \rightarrow M^{coH}$ , where  $M^{coH}$  denotes the subspace of coinvariants. So  $\mathcal{M}_B$  has an  $\mathcal{M}^H$ -module category structure.

**3.2. Duality for fusion categories and categorical Morita equivalence.** Let  $\mathcal{A}$  be a fusion category and let  $\mathcal{M}$  be an indecomposable left  $\mathcal{A}$ -module category. The category  $\mathcal{A}_{\mathcal{M}}^*$  of  $\mathcal{A}$ -module endofunctors of  $\mathcal{M}$  has a tensor category structure with a tensor product given by the composition of functors and the unit object being the identity functor. It is also a rigid category with the duals of a functor being its adjoints (thanks to the rigidity of  $\mathcal{A}$  an adjoint of an  $\mathcal{A}$ -module functor has a natural  $\mathcal{A}$ -module functor structure).

It was shown in [11, Theorem 2.18] that  $\mathcal{A}_{\mathcal{M}}^*$  is a fusion category. The category  $\mathcal{A}_{\mathcal{M}}^*$  is called the *dual* category of  $\mathcal{A}$  with respect to  $\mathcal{M}$ . Furthermore,  $\mathcal{M}$  has

a natural structure of an  $\mathcal{A}_{\mathcal{M}}^*$ -module category and there is a canonical tensor equivalence

$$(\mathcal{A}_{\mathcal{M}}^*)_{\mathcal{M}}^* \cong \mathcal{A}.$$

The Frobenius-Perron dimension is invariant under duality, i.e.,

$$(33) \quad \text{FPdim}(\mathcal{A}) = \text{FPdim}(\mathcal{A}_{\mathcal{M}}^*).$$

**Example 3.10.** Let  $A$  be a separable algebra in  $\mathcal{A}$  and let  $\mathcal{M}$  be the category of right  $A$ -modules in  $\mathcal{A}$ . Then  $(\mathcal{A}_{\mathcal{M}}^*)^{\text{op}}$  is tensor equivalent to the category of  $A$ -bimodules in  $\mathcal{A}$ . The tensor product of the latter category is  $\otimes_A$  and the unit object is the regular  $A$ -module.

**Definition 3.11.** Let  $\mathcal{A}, \mathcal{B}$  be fusion categories. We say that  $\mathcal{A}$  and  $\mathcal{B}$  are *categorically Morita equivalent* if there is an  $\mathcal{A}$ -module category  $\mathcal{M}$  such that  $\mathcal{B} \cong (\mathcal{A}_{\mathcal{M}}^*)^{\text{op}}$ .

It was shown in [29] that categorical Morita equivalence is indeed an equivalence relation.

**Remark 3.12.** The class of integral fusion categories (see Definition 2.24) is closed under Morita equivalence [11, Theorem 8.35].

Given a pair of left  $\mathcal{A}$ -module categories let  $\text{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$  denote the category of  $\mathcal{A}$ -module functors from  $\mathcal{M}$  to  $\mathcal{N}$ . In particular,  $\mathcal{A}_{\mathcal{M}}^* = \text{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ . The assignment

$$\mathcal{N} \mapsto \text{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$$

defines a 2-equivalence between the 2-category of left  $\mathcal{A}$ -module categories and that of right  $\mathcal{A}_{\mathcal{M}}^*$ -module categories [15, 28]. This explains the observation we made in Remark 3.6.

Below we give examples of categorical Morita equivalence.

**Example 3.13.** Any fusion category  $\mathcal{A}$  can be viewed as the regular left module category over itself. It is easy to see that in this case left  $\mathcal{A}$ -module functors are precisely functors of the right multiplication by objects of  $\mathcal{A}$ , whence  $\mathcal{A}_{\mathcal{A}}^* = \mathcal{A}^{\text{op}}$ .

**Example 3.14.** Any fusion category  $\mathcal{A}$  can be viewed as an  $\mathcal{A} \boxtimes \mathcal{A}^{\text{op}}$ -module category with the actions of  $\mathcal{A}$  given by left and right tensor multiplications. The dual category  $(\mathcal{A} \boxtimes \mathcal{A}^{\text{op}})_{\mathcal{A}}^*$  is the center of  $\mathcal{A}$ , see Section 4.3 below.

**Example 3.15.** Let  $G$  be a finite group and let  $\mathcal{A} = \text{Vec}_G$  be the category of  $G$ -graded vector spaces. The category  $\text{Vec}$  is a  $\text{Vec}_G$ -module category via the forgetful tensor functor  $\text{Vec}_G \rightarrow \text{Vec}$ . Let us determine the dual category  $(\text{Vec}_G)_{\text{Vec}}^*$ . Unfolding the definition of a module functor, we see that a  $\text{Vec}_G$ -module endofunctor  $F : \text{Vec} \rightarrow \text{Vec}$  is determined by a vector space  $V := F(\mathbb{k})$  and a collection of isomorphisms

$$\pi_g \in \text{Hom}_{\text{Vec}}(F(\delta_g \otimes k), \delta_g \otimes F(\mathbb{k})) = \text{End}_{\mathbb{k}}(V), \quad g \in G.$$

It follows from axiom (28) in Definition 3.2 of module functor that the map

$$g \mapsto \pi_g : G \rightarrow GL(V)$$

is a representation of  $G$  on  $V$ . Conversely, any such representation determines a  $\text{Vec}_G$ -module endofunctor of  $\text{Vec}$ . It is easy to check that homomorphisms of representations are precisely morphisms between the corresponding module functors. Thus,  $(\text{Vec}_G)_{\text{Vec}}^* \cong \text{Rep}(G)$ , i.e., the categories  $\text{Vec}_G$  and  $\text{Rep}(G)$  are categorically Morita equivalent.



**Example 3.16.** This is a generalization of the previous example. Let  $H$  be a finite-dimensional Hopf algebra. The fiber functor  $\text{Rep}(H) \rightarrow \text{Vec}$  makes  $\text{Vec}$  a  $\text{Rep}(H)$ -module category and  $\text{Rep}(H)^*_{\text{Vec}} \cong \text{Rep}(H^*)$ . Thus,  $\text{Rep}(H)$  and  $\text{Rep}(H^*)$  are categorically Morita equivalent. This means that categorical duality extends the notion of Hopf algebra duality.

**Example 3.17.** Let  $G$  be a finite group and let  $g \mapsto T_g$  be an action of  $G$  on a fusion category  $\mathcal{A}$ . The forgetful tensor functor  $\mathcal{A}^G \rightarrow \mathcal{A}$  turns  $\mathcal{A}$  into an  $\mathcal{A}^G$ -module fusion category. The dual category  $(\mathcal{A}^G)^*_{\mathcal{A}}$  is the *crossed product* category  $\mathcal{A} \rtimes G$  defined as follows. As an Abelian category  $\mathcal{A} \rtimes G = \mathcal{A} \boxtimes \text{Vec}_G$ . The tensor product is given by

$$(34) \quad (X \boxtimes \delta_g) \otimes (Y \boxtimes \delta_h) := (X \otimes T_g(Y)) \boxtimes \delta_{gh}, \quad X, Y \in \mathcal{A}, \quad g, h \in G.$$

The unit object is  $\mathbf{1} \boxtimes \delta_e$  and the associativity and unit constraints come from those of  $\mathcal{A}$ .

Note that  $\mathcal{C} \rtimes G$  is a  $G$ -graded fusion category,

$$\mathcal{A} \rtimes G = \bigoplus_{g \in G} (\mathcal{A} \rtimes G)_g, \quad \text{where } (\mathcal{A} \rtimes G)_g = \mathcal{C} \otimes (\mathbf{1} \boxtimes \delta_g).$$

In particular,  $\mathcal{A} \rtimes G$  contains  $\mathcal{A} = \mathcal{A} \otimes (\mathbf{1} \boxtimes \delta_e)$  as a fusion subcategory.

In the case when  $\mathcal{A} = \text{Vec}$  we recover the duality of Example 3.15.

**3.3. (Weakly) group-theoretical fusion categories.** In this Section we use categorical Morita equivalence to introduce two classes of categories important for classification of fusion categories of integer Frobenius-Perron dimension.

**Definition 3.18.** A fusion category is called *group-theoretical* if it is categorically Morita equivalent to a pointed fusion category.

In other words, a group-theoretical fusion category is equivalent to  $(\text{Vec}_G^\omega)^*_{\mathcal{M}(L, \psi)}$ , where  $\mathcal{M}(L, \psi)$  is the  $\text{Vec}_G^\omega$ -module category from Example 3.7.

**Example 3.19.** The category  $(\text{Vec}_G^\omega)^*_{\mathcal{M}(L, \psi)}$  can be described quite explicitly in terms of finite groups and their cohomology as the category of  $A(L, \psi)$ -bimodules in  $\text{Vec}_G^\omega$ , where  $A(L, \psi)$  is the algebra introduced in (31) (see [35, Proposition 3.1]).

For example, simple objects of  $(\text{Vec}_G^\omega)^*_{\mathcal{M}(L, \psi)}$  can be described as follows.

For any  $g \in G$  the group  $L^g := L \cap gLg^{-1}$  has a well-defined 2-cocycle

$$\begin{aligned} \psi^g(h, h') : &= \psi(h, h')\psi(g^{-1}h'^{-1}g, g^{-1}h^{-1}g)\omega(hh'g, g^{-1}h'^{-1}g, g^{-1}h^{-1}g)^{-1} \\ &\quad \times \omega(h, h', g)\omega(h, h'g, g^{-1}h'^{-1}g), \quad h, h' \in L^g. \end{aligned}$$

One can check that irreducible  $A(L, \psi)$ -bimodules in  $\text{Vec}_G^\omega$  are parameterized by pairs  $(Z, \pi)$ , where  $Z$  is a double  $L$ -coset in  $G$  and  $\pi$  is an irreducible projective representation of  $L^g$  with the Schur multiplier  $\psi^g$ ,  $g \in Z$ .

**Example 3.20.** Let  $G, L$  be groups and let  $H$  be a semisimple Hopf algebra that fits into an exact sequence

$$1 \rightarrow \mathbb{k}^G \rightarrow H \rightarrow \mathbb{k}L \rightarrow 1,$$

where  $\mathbb{k}^G$  is the commutative Hopf algebra of functions on  $G$  and  $\mathbb{k}L$  is the cocommutative group Hopf algebra of  $L$  (i.e.,  $H$  is an extension of  $\mathbb{k}L$  by  $\mathbb{k}^G$ ). It was shown in [30] that  $\text{Rep}(H)$  is a group-theoretical fusion category.

**Remark 3.21.** The class of group-theoretical fusion categories is not closed under equivariantizations [17, 33]. In particular, the class of Hopf algebras with group-theoretical representation categories is not closed under Hopf algebra extensions.

The smallest example of a semisimple Hopf algebra whose representation category is not group-theoretical has dimension 36 [33].

**Remark 3.22.** In view of Remark 3.12, group-theoretical categories are integral.

Recall that the notion of a nilpotent fusion category was defined in Section 2.3. The following definition was given in [12].

**Definition 3.23.** A fusion category is *weakly group-theoretical* if it is categorically Morita equivalent to a nilpotent fusion category. A fusion category is *solvable* if it is categorically Morita equivalent to a cyclically nilpotent fusion category.

Here is a list of properties of solvable categories (see [12, Proposition 4.4]).

- Proposition 3.24.**
- (i) *The class of solvable categories is closed under taking extensions and equivariantizations by solvable groups, Morita equivalent categories, tensor products, subcategories and component categories of quotient categories.*
  - (ii) *The categories  $\text{Vec}_{G,\omega}$  and  $\text{Rep}(G)$  are solvable if and only if  $G$  is a solvable group.*
  - (iii) *A braided nilpotent fusion category is solvable.*
  - (iv) *A solvable fusion category  $\mathcal{A} \neq \text{Vec}$  contains a nontrivial invertible object.*

**Lemma 3.25.** *Let  $G$  be a finite group, let  $\mathcal{A}$  be a  $G$ -extension of a fusion category  $\mathcal{A}_0$ , and let  $\mathcal{B}_0$  be a fusion category Morita equivalent to  $\mathcal{A}_0$ . There exists a  $G$ -extension  $\mathcal{B}$  of  $\mathcal{B}_0$  which is Morita equivalent to  $\mathcal{A}$ .*

*Proof.* The proof is taken from [12, Lemma 3.4]. Let  $A$  be an algebra in  $\mathcal{A}_0$  such that  $\mathcal{B}_0$  is equivalent to the category of  $A$ -bimodules in  $\mathcal{A}_0$ . Let  $\mathcal{B}$  be the category of  $A$ -bimodules in  $\mathcal{A}$  (we can view  $A$  as an algebra in  $\mathcal{A}$  since  $\mathcal{A}_0 \subset \mathcal{A}$ ). Then  $\mathcal{B}$  inherits the  $G$ -grading, thanks to  $A$  being in the trivial component of the  $G$ -graded fusion category  $\mathcal{A}$ . By construction,  $\mathcal{B}$  is categorically Morita equivalent to  $\mathcal{A}$ .  $\square$

**Proposition 3.26.** *The class of weakly group-theoretical fusion closed is closed under extensions and equivariantizations.*

*Proof.* In view of Example 3.17 it is enough to prove the assertion about extensions. Let  $\mathcal{A}$  be a  $G$ -extension of a weakly group-theoretical fusion category  $\mathcal{A}_0$ . Let  $\mathcal{B}_0$  be a nilpotent fusion category Morita equivalent to  $\mathcal{A}_0$ . Then by Lemma 3.25 there exists a nilpotent category  $\mathcal{B}$  Morita equivalent to  $\mathcal{A}$ , i.e.,  $\mathcal{A}$  is weakly group-theoretical.  $\square$

**Remark 3.27.** Proposition 3.26 and the fact that the class of weakly group-theoretical categories is closed under taking subcategories and quotient categories were proved in [12, Proposition 4.1].

**Remark 3.28.** Since the Frobenius-Perron dimension of a fusion category is invariant under categorical Morita equivalence, cf. (33), we have  $\text{FPdim}(\mathcal{A}) \in \mathbb{Z}$  for every weakly group-theoretical fusion category  $\mathcal{A}$ .

**Remark 3.29.** Let  $\mathcal{A}$  be a weakly group-theoretical fusion category. The following Frobenius property of  $\mathcal{A}$  was established in [12, Theorem 1.5]: for every simple object  $X$  of  $\mathcal{A}$  the ratio  $\text{FPdim}(\mathcal{A})/\text{FPdim}(X)$  is an algebraic integer.

**Remark 3.30.** Because of recursive nature of the definition of a nilpotent fusion category it is usually not possible to describe weakly group-theoretical categories as explicitly as group-theoretical ones, cf. Example 3.19. On the other hand, there is a classification of module categories over a given graded fusion category in terms of module categories over its trivial component [16, 24]. So, in principle, weakly group-theoretical fusion categories can be described in terms of finite groups and their cohomology.

**Remark 3.31.** The approach to classification of fusion categories used in this paper consists of showing that categories of a given Frobenius-Perron dimension are weakly group-theoretical. We *do not* attempt to classify weakly group-theoretical categories of an arbitrary finite dimension (indeed, this would include, as a special case, classification of finite groups).

#### 4. BRAIDED FUSION CATEGORIES

The notion of a braiding was introduced by A. Joyal and R. Street in [19].

##### 4.1. Definitions and examples.

**Definition 4.1.** A *braiding* on a fusion category  $\mathcal{C}$  is a natural isomorphism

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \quad X, Y \in \mathcal{C},$$

called a *commutativity constraint*, such that the following hexagon diagrams (35)

$$\begin{array}{ccccc}
 & & X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\
 & \nearrow a_{X,Y,Z} & & & \searrow a_{Y,Z,X} \\
 (X \otimes Y) \otimes Z & & & & Y \otimes (Z \otimes X) \\
 & \searrow c_{X,Y} \otimes \text{id}_Z & & & \nearrow \text{id}_Y \otimes c_{X,Z} \\
 & & (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z)
 \end{array}$$

and

(36)

$$\begin{array}{ccccc}
 & & (X \otimes Y) \otimes Z & \xrightarrow{c_{X \otimes Y,Z}} & Z \otimes (X \otimes Y) \\
 & \nearrow a_{X,Y,Z}^{-1} & & & \searrow a_{Z,X,Y}^{-1} \\
 X \otimes (Y \otimes Z) & & & & (Z \otimes X) \otimes Y \\
 & \searrow \text{id}_X \otimes c_{Y,Z} & & & \nearrow c_{X,Z} \otimes \text{id}_Y \\
 & & X \otimes (Z \otimes Y) & \xrightarrow{a_{X,Z,Y}^{-1}} & (X \otimes Z) \otimes Y
 \end{array}$$

are commutative for all objects  $X, Y, Z$  in  $\mathcal{C}$ .

For a braided fusion category  $\mathcal{C}$  with the braiding  $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$  let  $\mathcal{C}^{\text{rev}}$  denote the *reverse* category that coincides with  $\mathcal{C}$  as a fusion category and has braiding  $\tilde{c}_{X,Y} := c_{Y,X}^{-1}$ .

**Definition 4.2.** Let  $\mathcal{C}^1$  and  $\mathcal{C}^2$  be braided tensor categories whose braidings are denoted  $c^1$  and  $c^2$ , respectively. A tensor functor  $(F, J)$  from  $\mathcal{C}^1$  to  $\mathcal{C}^2$  is called *braided* if the following diagram commutes:

$$(37) \quad \begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{c_{F(X), F(Y)}^2} & F(Y) \otimes F(X) \\ J_{X,Y} \downarrow & & \downarrow J_{Y,X} \\ F(X \otimes Y) & \xrightarrow{F(c_{X,Y}^1)} & F(Y \otimes X) \end{array}$$

for all object  $X, Y$  in  $\mathcal{C}^1$ .

Note that a tensor functor is a functor with an additional *structure*. For a tensor functor to be braided is a *property*.

Let  $G$  be an Abelian group. By a *quadratic form* on  $G$  (with values in  $\mathbb{k}^\times$ ) we will mean a map  $q : G \rightarrow \mathbb{k}^\times$  such that  $q(g) = q(g^{-1})$  and the symmetric function

$$(38) \quad b(g, h) := \frac{q(gh)}{q(g)q(h)}$$

is bimultiplicative, i.e.,  $b(g_1g_2, h) = b(g_1, h)b(g_2, h)$  for all  $g, g_1, g_2, h \in G$ . We will say that  $q$  is *non-degenerate* if the associated bicharacter  $b$  is non-degenerate.

The simplest way to construct a quadratic form on  $G$  is to start with a bicharacter  $B : G \times G \rightarrow \mathbb{k}^\times$  and set

$$(39) \quad q(g) := B(g, g), \quad g \in G.$$

**Definition 4.3.** A *pre-metric group* is a pair  $(G, q)$  where  $G$  is a finite Abelian group and  $q : G \rightarrow \mathbb{k}^\times$  is a quadratic form. A *metric group* is a pre-metric group such that  $q$  is non-degenerate.

The relation between braided fusion categories and pre-metric groups is as follows. Let  $\mathcal{C}$  be a pointed braided fusion category. Then  $\mathcal{C} = \text{Vec}_G^\omega$  for some Abelian group  $G$ . Define a map  $q : G \rightarrow \mathbb{k}^\times$  by

$$q(g) = c_{\delta_g, \delta_g} \in \text{Aut}(\delta_g \otimes \delta_g) = \mathbb{k}^\times.$$

It is easy to see that  $q : G \rightarrow \mathbb{k}^\times$  is a quadratic form. We thus have a functor:

$$F : (\text{pointed braided fusion categories}) \rightarrow (\text{pre-metric groups}).$$

It was shown by Joyal and Street in [19] that this functor is an equivalence. Under this equivalence, braided tensor functors correspond to orthogonal (i.e., quadratic form preserving) homomorphisms.

We will denote by  $\mathcal{C}(G, q)$  the braided fusion category associated to the pre-metric group  $(G, q)$ .

**Remark 4.4.** When  $q$  is determined by a bicharacter  $B$  as in (39) we have  $\mathcal{C}(G, q) = \text{Vec}_G$  as a fusion category with the braiding given by

$$c_{\delta_g, \delta_h} = B(g, h) \text{id}_{\delta_{gh}}.$$

**Definition 4.5.** A *quasi-triangular structure* on a Hopf algebra  $H$  is an invertible element  $R \in H \otimes H$  such that for all  $x \in H$ ,

$$(40) \quad R\Delta(x) = \Delta^{op}(x)R,$$

where  $\Delta^{op}$  denotes the opposite comultiplication, and the following relations are satisfied (in  $H \otimes H \otimes H$ ):

$$(41) \quad (\Delta \otimes \text{id})(R) = R^{13}R^{23}$$

$$(42) \quad (\text{id} \otimes \Delta)(R) = R^{13}R^{12}$$

A Hopf algebra  $H$  equipped with a quasi-triangular structure is said to be a *quasi-triangular Hopf algebra*.

Here for  $R = \sum_i r_i \otimes r'_i$  we write  $R^{12} = \sum_i r_i \otimes r'_i \otimes 1 \in H \otimes H \otimes H$ , etc.

Given a semisimple quasi-triangular Hopf algebra  $H$  one turns  $\text{Rep}(H)$  into a braided fusion category by setting

$$(43) \quad c_{V \otimes W} : V \otimes W \xrightarrow{\sim} W \otimes V : v \otimes w \mapsto R^{21}(w \otimes v),$$

for all representations  $V, W$  of  $H$  and  $v \in V, w \in W$ .

Note that the axiom (40) means that the map (43) is a morphism in  $\text{Rep}(H)$  and axioms (41) and (42) are equivalent to  $c$  satisfying the hexagon axioms (35) and (36). Conversely, any braiding on  $\text{Rep}(H)$  is determined by a quasi-triangular structure.

**4.2. Symmetric and Tannakian subcategories.** A braided fusion category  $\mathcal{C}$  is called *symmetric* if  $c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}$  for all objects  $X, Y \in \mathcal{C}$ ; in this case the braiding  $c$  is also called *symmetric*.

**Example 4.6.** The category  $\text{Rep}(G)$  of representations of a finite group  $G$  equipped with its standard symmetric braiding  $c_{X,Y}(x \otimes y) := y \otimes x$  is an example of a symmetric fusion category. Deligne [7] proved that any symmetric fusion category is equivalent to a “super” generalization of  $\text{Rep}(G)$ . Namely, let  $G$  be a finite group and let  $z \in G$  be a central element such that  $z^2 = 1$ . Then the fusion category  $\text{Rep}(G)$  has a braiding  $c'_{X,Y}$  defined as follows:

$$c'_{X,Y}(x \otimes y) = (-1)^{mn} y \otimes x \text{ if } x \in X, y \in Y, zx = (-1)^m x, zy = (-1)^n y.$$

Let  $\text{Rep}(G, z)$  denote the fusion category  $\text{Rep}(G)$  equipped with the above braiding. Equivalently,  $\text{Rep}(G, z)$  can be described as a full subcategory of the category of super-representations of  $G$ ; namely,  $\text{Rep}(G, z)$  consists of those super-representations  $V$  on which  $z$  acts by the *parity automorphism* (i.e.,  $zv = v$  if  $v \in V$  is even and  $zv = -v$  if  $v \in V$  is odd).

For example, let  $G = \mathbb{Z}/2\mathbb{Z}$  and  $z$  be the nontrivial element of  $G$ . Then  $\text{Rep}(G, z)$  is the category  $\mathbf{sVec}$  of super-vector spaces.

A symmetric fusion category  $\mathcal{C}$  is said to be *Tannakian* if there exists a finite group  $G$  such that  $\mathcal{C}$  is equivalent to  $\text{Rep}(G)$  as a braided fusion category. It is proved in [7] that  $\mathcal{C}$  is Tannakian if and only if it admits a *braided fiber functor*, i.e., a braided tensor functor  $\mathcal{C} \rightarrow \mathbf{Vec}$ .

The canonical fiber functor  $\text{Rep}(G) \rightarrow \mathbf{Vec}$  is nothing but the functor forgetting the  $G$ -module structure.

Note that for a symmetric category  $\mathcal{C}$  its dimension  $\text{FPdim}(\mathcal{C})$  is always an integer (more precisely,  $\text{FPdim}(\text{Rep}(G, z)) = |G|$ ). In particular, if  $\text{FPdim}(\mathcal{C})$  is *odd* then  $\mathcal{C}$  is automatically Tannakian.

**Remark 4.7.** Let  $\mathcal{C} = \text{Rep}(G, z)$  be a symmetric category. Then  $\text{Rep}(G/\langle z \rangle)$  is a Tannakian subcategory of  $\mathcal{C}$ . In particular, a symmetric fusion category  $\mathcal{C} \not\cong \mathbf{sVec}$  contains a non-trivial Tannakian subcategory.

**Example 4.8.** The pointed braided fusion category  $\mathcal{C}(G, q)$  associated to the pre-metric group  $(G, q)$  is symmetric if and only if  $q$  is a homomorphism. The category  $\mathcal{C}(G, q)$  is Tannakian if and only if  $q = 1$ .

**4.3. The Drinfeld center construction.** We now give a construction which assigns to every fusion category  $\mathcal{A}$  a braided fusion category  $\mathcal{Z}(\mathcal{A})$ , called the *center* of  $\mathcal{A}$ .

Explicitly, the objects of  $\mathcal{Z}(\mathcal{A})$  are pairs  $(X, \gamma)$ , where  $X$  is an object of  $\mathcal{A}$  and

$$(44) \quad \gamma = \{\gamma_V : V \otimes X \xrightarrow{\sim} X \otimes V\}_{V \in \mathcal{A}}$$

is a natural family of isomorphisms, called *half-braidings*, making the following diagram commutative:

$$(45) \quad \begin{array}{ccccc} & & V \otimes (X \otimes U) & \xrightarrow{a_{V,X,U}^{-1}} & (V \otimes X) \otimes U \\ & \nearrow \text{id}_V \otimes \gamma_U & & & \searrow \gamma_V \otimes \text{id}_U \\ V \otimes (U \otimes X) & & & & (X \otimes V) \otimes U \\ & \searrow a_{V,U,X}^{-1} & & & \nearrow a_{X,V,U}^{-1} \\ & & (V \otimes U) \otimes X & \xrightarrow{\gamma_V \otimes U} & X \otimes (V \otimes U) \end{array}$$

The center has a canonical braiding given by

$$(46) \quad c_{(X,\gamma),(X',\gamma')} = \gamma_{X'} : (X, \gamma) \otimes (X', \gamma') \xrightarrow{\sim} (X', \gamma') \otimes (X, \gamma).$$

Furthermore, there is an obvious forgetful tensor functor:

$$(47) \quad F : \mathcal{Z}(\mathcal{A}) \mapsto \mathcal{A} : (X, \gamma) \mapsto X.$$

We have

$$(48) \quad \text{FPdim}(\mathcal{Z}(\mathcal{A})) = \text{FPdim}(\mathcal{A})^2 \quad \text{and} \quad \dim(\mathcal{Z}(\mathcal{A})) = \dim(\mathcal{A})^2,$$

where the Frobenius-Perron dimension  $\text{FPdim}(\mathcal{A})$  and categorical dimension  $\dim(\mathcal{A})$  were introduced in (17) and (18).

Let  $\mathcal{C}$  be a braided fusion category with braiding  $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ . There are natural braided embeddings  $\mathcal{C}, \mathcal{C}^{\text{rev}} \hookrightarrow \mathcal{Z}(\mathcal{C})$  given by

$$X \mapsto (X, c_{-,X}) \quad \text{and} \quad X \mapsto (X, \tilde{c}_{-,X}).$$

They combine into a braided tensor functor

$$(49) \quad G : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}).$$

We say that  $\mathcal{C}$  is *factorizable* if the functor (49) is an equivalence.

**Example 4.9.** Let  $A$  be a finite Abelian group. There is canonical quadratic form

$$(50) \quad q : A \oplus A^* \rightarrow \mathbb{k}^\times : (a, \phi) \mapsto \phi(a), \quad a \in A, \phi \in A^*.$$

We have  $\mathcal{Z}(\text{Vec}_A) \cong \mathcal{C}(A \oplus A^*, q)$ .

**Example 4.10.** More generally, let  $G$  be a finite group. The category  $\mathcal{Z}(\text{Vec}_G)$  is equivalent to the category of  $G$ -equivariant vector bundles on  $G$ , cf. Example 2.22(iii). Here  $G$  acts on itself by conjugation. Explicitly, a  $G$ -equivariant

vector bundle is a graded vector space  $V = \bigoplus_{g \in G} V_g$  along with a collection of isomorphisms  $\phi_{x,g} : V_g \rightarrow V_{xgx^{-1}}$  satisfying the compatibility condition

$$\phi_{x, ygy^{-1}} \phi_{y,g} = \phi_{xy,g},$$

for all  $x, y, g \in G$ . Note that  $\mathcal{Z}(\text{Vec}_G)$  contains a subcategory  $\text{Rep}(G)$  as the bundles supported on the identity element of  $G$ .

**Example 4.11.** Let  $H$  be a semisimple Hopf algebra. Then

$$\mathcal{Z}(\text{Rep}(H)) \cong \text{Rep}(D(H)),$$

where  $D(H)$  is the Drinfeld double of  $H$  [21].

#### 4.4. Ribbon fusion categories and traces.

**Definition 4.12.** A *pre-modular* fusion category is a braided fusion category equipped with a spherical structure.

Below we give an equivalent description of pre-modular categories.

**Definition 4.13.** A *twist* (or a *balanced transformation*) on a braided fusion category  $\mathcal{C}$  is  $\theta \in \text{Aut}(\text{id}_{\mathcal{C}})$  such that

$$(51) \quad \theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) c_{Y,X} c_{X,Y}$$

for all  $X, Y \in \mathcal{C}$ . A twist is called a *ribbon structure* if  $(\theta_X)^* = \theta_{X^*}$ . A fusion category with a ribbon structure is called a *ribbon category*.

**Remark 4.14.** The notion of a ribbon structure can be understood as a generalization of the notion of quadratic form. Indeed, let  $G$  be a finite Abelian group and  $b : G \times G \rightarrow \mathbb{k}^\times$  be a bilinear form. As explained in Section 4.1, it defines a braiding on  $\mathcal{C} = \text{Vec}_G$ . The corresponding quadratic form defines a ribbon structure on  $\mathcal{C}$ :

$$\theta_{\delta_x} = b(x, x) \text{id}_{\delta_x}, \quad x \in G.$$

Let us define a natural transformation  $u_X : X \rightarrow X^{**}$  as the composition

$$(52) \quad X \xrightarrow{\text{id}_X \otimes \text{coev}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{c_{X,X^*} \otimes \text{id}_{X^{**}}} X^* \otimes X \otimes X^{**} \xrightarrow{\text{ev}_X \otimes \text{id}_{X^{**}}} X^{**}.$$

Then  $u_X$  is an isomorphism satisfying the following balancing property:

$$(53) \quad u_X \otimes u_Y = u_{X \otimes Y} c_{Y,X} c_{X,Y}$$

or all  $X, Y \in \mathcal{C}$ .

Clearly, any natural isomorphism  $\psi_X : X \simeq X^{**}$  in a braided fusion category  $\mathcal{C}$  can be written as

$$(54) \quad \psi_X = u_X \theta_X.$$

for some  $\theta \in \text{Aut}(\text{id}_{\mathcal{C}})$ . It follows from (53) that  $\psi$  is a tensor isomorphism (i.e., a pivotal structure on  $\mathcal{C}$ ) if and only if  $\theta$  is a twist.

The above pivotal structure is spherical if and only if the corresponding twist  $\theta = \psi u^{-1}$  is a ribbon structure. Thus, a ribbon fusion category is the same thing as a pre-modular category.

**4.5.  $S$ -matrix of a pre-modular category.** Let  $\mathcal{C}$  be a pre-modular category with a spherical structure  $\psi$ . Let  $\mathcal{O}(\mathcal{C})$  denote the set of (isomorphism classes of) simple objects of  $\mathcal{C}$ . For all  $X, Y, Z \in \mathcal{O}(\mathcal{C})$  let  $N_{XY}^Z$  denote the multiplicity of  $Z$  in  $X \otimes Y$ .

We will identify the corresponding twist  $\theta \in \text{Aut}(\text{id}_{\mathcal{C}})$  with a collection of scalars  $\theta_X \in \mathbb{k}^\times$ ,  $X \in \mathcal{O}(\mathcal{C})$ . Let  $\text{Tr}$  and  $d$  denote the trace and dimension corresponding to  $\psi$ .

**Definition 4.15.** Let  $\mathcal{C}$  be a pre-modular category. The  $S$ -matrix of  $\mathcal{C}$  is defined by

$$(55) \quad S := (s_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})}, \quad \text{where} \quad s_{XY} = \text{Tr}(c_{Y,X} c_{X,Y}).$$

**Remark 4.16.** The  $S$ -matrix of  $\mathcal{C}$  is a symmetric  $n$ -by- $n$  matrix where  $n = |\mathcal{O}(\mathcal{C})|$  is the number of simple objects of  $\mathcal{C}$ . It satisfies  $s_{X^*Y^*} = s_{XY}$  for all  $X, Y \in \mathcal{O}(\mathcal{C})$ . We also have  $s_{X1} = s_{1X} = d_X$ .

**Definition 4.17.** ([40]) A pre-modular category  $\mathcal{C}$  is said to be *modular* if its  $S$ -matrix is non-degenerate.

**Example 4.18.** Let  $G$  be a finite Abelian group. Let  $q : G \rightarrow \mathbb{k}^\times$  be a quadratic form on  $G$  and let  $b : G \times G \rightarrow \mathbb{k}^\times$  be the associated symmetric bilinear form. The  $S$ -matrix of the corresponding pointed premodular category  $\mathcal{C}(G, q)$  (see Section 4.1) is  $\{b(g, h)\}_{g,h \in G}$ . Thus,  $\mathcal{C}(G, q)$  is modular if and only if  $q$  is non-degenerate.

Let  $\mathcal{C}$  be a pre-modular category.

**Proposition 4.19.** *We have*

$$(56) \quad s_{XY} = \theta_X^{-1} \theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^Z \theta_Z d_Z.$$

for all  $X, Y \in \mathcal{O}(\mathcal{C})$ .

*Proof.* Apply  $\text{Tr}$  to both sides of formula (51). The right hand side is equal to  $\theta_X \theta_Y s_{XY}$  while the left hand side is equal to

$$\begin{aligned} \text{Tr}(\theta_{X \otimes Y}) &= \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^Z \text{Tr}(\theta_Z \text{id}_Z) \\ &= \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^Z \theta_Z d_Z, \end{aligned}$$

where we used additivity of  $\text{Tr}$ . □

**Remark 4.20.** When  $\mathcal{C} = \mathcal{C}(G, q)$  the relation (56) between the twist and  $S$ -matrix of a premodular category generalizes the relation (38) between the quadratic form and associated bilinear form.

The elements of  $S$ -matrix satisfy the following *Verlinde formula* (see [3, Theorem 3.1.12], [27, Lemma 2.4 (iii)] for a proof):

$$(57) \quad s_{XY} s_{XZ} = d_X \sum_{W \in \mathcal{O}(\mathcal{C})} N_{YZ}^W s_{XW}, \quad X, Y, Z \in \mathcal{O}(\mathcal{C}).$$



**Remark 4.21.** Formula (57) can be interpreted as follows. For any fixed  $X \in \mathcal{O}(\mathcal{C})$  the map

$$(58) \quad h_X : Y \mapsto \frac{s_{XY}}{d_X}, \quad Y \in \mathcal{O}(\mathcal{C})$$

gives rise to a ring homomorphism  $K(\mathcal{C}) \rightarrow \mathbb{k}$  which we will also denote  $h_X$ . That is, simple objects of  $\mathcal{C}$  give rise to characters of the Grothendieck ring  $K(\mathcal{C})$ . We have  $h_1(Y) = d_Y$ .

The characters satisfy the following orthogonality relation:

$$(59) \quad \sum_{X \in \mathcal{O}(\mathcal{C})} h_Y(X) h_Z(X^*) = 0 \quad \text{for } Y \not\cong Z.$$

The following result is due to Anderson, Moore, and Vafa [1, 42].

**Theorem 4.22.** *Let  $\mathcal{C}$  be a premodular category. Let  $\theta$  be the twist of  $\mathcal{C}$ . Then  $\theta_X$  is a root of unity for all  $X \in \mathcal{C}$ .*

**4.6. Modular categories.** The categorical dimension of a fusion category  $\mathcal{C}$  was defined in (18). When  $\mathcal{C}$  is premodular we have

$$(60) \quad \dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} d_X^2,$$

where  $d$  is the dimension associated to the spherical structure of  $\mathcal{C}$ .

Let  $E = \{E_{XY}\}_{X,Y \in \mathcal{O}(\mathcal{C})}$  be the square matrix such that  $E_{XY} = 1$  if  $X = Y^*$  and  $E_{XY} = 0$  otherwise.

**Proposition 4.23.** *Let  $\mathcal{C}$  be a modular category and  $S$  be its  $S$ -matrix. Then  $S^2 = \dim(\mathcal{C})E$ .*

*Proof.* Since  $S$  is non-degenerate, the equality  $h_Y = h_Z$  for  $Y, Z \in \mathcal{O}(\mathcal{C})$  holds if and only if  $Y = Z$ , where  $h_Y : K(\mathcal{C}) \rightarrow \mathbb{k}$  are the characters defined in (58).

Suppose  $Y \neq Z$ . Using (59) we have

$$\sum_{X \in \mathcal{O}(\mathcal{C})} s_{XY} s_{XZ^*} = \sum_{X \in \mathcal{O}(\mathcal{C})} s_{XY} s_{X^*Z} = 0$$

It remains to check that  $\sum_{X \in \mathcal{O}(\mathcal{C})} s_{XY} s_{XY^*} = \dim(\mathcal{C})$  for all  $Y \in \mathcal{O}(\mathcal{C})$ . We compute

$$\begin{aligned} \sum_{X \in \mathcal{O}(\mathcal{C})} s_{XY} s_{XY^*} &= \sum_{X \in \mathcal{O}(\mathcal{C})} d_X s_{XW} \sum_{W \in \mathcal{O}(\mathcal{C})} N_{YY^*}^W \\ &= \dim(\mathcal{C}) N_{YY^*}^1 = \dim(\mathcal{C}). \end{aligned}$$

Here the first equality is (57). The second equality is a consequence of orthogonality of characters (59), since

$$\sum_{X \in \mathcal{O}(\mathcal{C})} d_X s_{XW} = d_W \sum_{X \in \mathcal{O}(\mathcal{C})} d_X h_W(X^*)$$

and the latter expression is equal to  $\dim(\mathcal{C})$  if  $W = 1$  and 0 otherwise.  $\square$

**Corollary 4.24.** *Let  $\mathcal{C}$  be a modular category. For all objects  $Y, Z, W \in \mathcal{O}(\mathcal{C})$  we have*

$$(61) \quad \sum_{X \in \mathcal{O}(\mathcal{C})} \frac{s_{XY} s_{XZ} s_{XW^*}}{d_X} = \dim(\mathcal{C}) N_{YZ}^W.$$

Thus, the  $S$ -matrix determines the fusion rules of  $\mathcal{C}$ .  
For any  $Z \in \mathcal{O}(\mathcal{C})$  define the following square matrices:

$$D^Z := \left( \delta_{XY} \frac{s_{XZ}}{d_X} \right)_{X,Y \in \mathcal{O}(\mathcal{C})} \quad \text{and} \quad N^Z = (N_{YZ}^W)_{Y,W \in \mathcal{O}(\mathcal{C})}.$$

**Corollary 4.25.** *Let  $\mathcal{C}$  be a modular category with the  $S$ -matrix  $S$ . Then  $D^Z = S^{-1}N^Z S$  for all  $Z \in \mathcal{O}(\mathcal{C})$ , i.e., conjugation by the  $S$ -matrix diagonalizes the fusion rules of  $\mathcal{C}$ .*

**Proposition 4.26.** *Let  $\mathcal{C}$  be a modular category and let  $X \in \mathcal{O}(\mathcal{C})$ . Then  $\frac{\dim(\mathcal{C})}{d_X^2}$  is an algebraic integer.*

*Proof.* We compute, using Proposition 4.23:

$$(62) \quad \frac{\dim(\mathcal{C})}{d_X^2} = \sum_{Y \in \mathcal{O}(\mathcal{C})} \frac{s_{XY}}{d_X} \frac{s_{XY^*}}{d_X} = \sum_{Y \in \mathcal{O}(\mathcal{C})} h_Y(X) h_{Y^*}(X),$$

where  $h_Y$ ,  $Y \in \mathcal{O}(\mathcal{C})$ , are characters of  $K(\mathcal{C})$  defined in (58). Since  $h_Y(X)$  is an eigenvalue of the integer matrix  $N^X$ , it is an algebraic integer. Hence, the right hand side of (62) is an algebraic integer.  $\square$

**4.7. Modular group representation and Galois action.** Modular categories have important arithmetic features that we describe next.

The *modular group* is, by definition, the group  $\Gamma := SL_2(\mathbb{Z})$  of  $2 \times 2$  matrices with integer entries and determinant 1.

It is known that  $\Gamma$  is generated by two matrices

$$(63) \quad \mathfrak{s} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{t} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let  $\mathcal{C}$  be a modular category. It turns out that  $\mathcal{C}$  gives rise to a projective representation of  $\Gamma$ . This justifies the terminology. Namely, let  $S = (s_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})}$  be the  $S$ -matrix of  $\mathcal{C}$  and let  $T = (t_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})}$  be a diagonal matrix with entries  $t_{XY} = \delta_{X,Y} \theta_X$ .

The assignments

$$(64) \quad \mathfrak{s} \mapsto \frac{1}{\sqrt{\dim(\mathcal{C})}} S \quad \text{and} \quad \mathfrak{t} \mapsto T$$

define a projective representation  $\rho : \Gamma \rightarrow GL_{|\mathcal{O}(\mathcal{C})|}(\mathbb{K})$ . When  $\mathcal{C}$  is the center of a fusion category this representation is linear.

Let  $N$  denote the order of  $T$ . It was shown in [32] that the kernel of  $\rho$  is a congruence subgroup of level  $N$  (i.e.,  $\text{Ker}(\rho)$  contains the kernel of the natural group homomorphism  $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$ ). For Hopf algebras this result was established in [39].

The entries of  $S$  and  $T$  are integers in  $\mathbb{Q}[\xi_N]$ , where  $\xi_N$  is a primitive  $N$ th root of unity [5, 6]. Furthermore, matrices in the image of  $\rho$  have the following remarkable property with respect to the Galois group  $\text{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})$  (see [8] and references therein). For every  $\sigma \in \text{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})$  the matrix  $G_\sigma := \sigma(S)S^{-1}$  is a signed permutation matrix and

$$\sigma^2(\rho(M)) = G_\sigma \rho(M) G_\sigma^{-1}$$

for all  $M \in \Gamma$ . In particular, there is a permutation  $\tilde{\sigma}$  of  $\mathcal{O}(\mathcal{C})$  such that

$$\sigma(s_{XY}) = \pm s_{X\tilde{\sigma}(Y)} \quad \text{and} \quad \sigma^2(\theta_X) = \pm \theta_{\tilde{\sigma}(X)}$$

for all  $X, Y \in \mathcal{O}(\mathcal{C})$ .

**4.8. Centralizers and non-degeneracy.** Recall from [27] that objects  $X$  and  $Y$  of a braided fusion category  $\mathcal{C}$  are said to *centralize* each other if

$$(65) \quad c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}.$$

The *centralizer*  $\mathcal{D}'$  of a fusion subcategory  $\mathcal{D} \subset \mathcal{C}$  is defined to be the full subcategory of objects of  $\mathcal{C}$  that centralize each object of  $\mathcal{D}$ . It is easy to see that  $\mathcal{D}'$  is a fusion subcategory of  $\mathcal{C}$ . Clearly,  $\mathcal{D}$  is symmetric if and only if  $\mathcal{D} \subset \mathcal{D}'$ .

**Definition 4.27.** We will say that a braided fusion category  $\mathcal{C}$  is *non-degenerate* if  $\mathcal{C}' = \text{Vec}$ .

For a fusion subcategory  $\mathcal{D}$  of a non-degenerate braided fusion category  $\mathcal{C}$  one has the following properties, see [27] and [10, Theorems 3.10, 3.14]:

$$(66) \quad \mathcal{D}'' = \mathcal{D},$$

$$(67) \quad \text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C}).$$

Furthermore, if  $\mathcal{D}$  is non-degenerate, then

$$(68) \quad \mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{D}'.$$

**Example 4.28.** Let  $\mathcal{C}$  be a non-degenerate braided fusion category. Let  $\mathcal{C}_{ad}$  be the adjoint subcategory of  $\mathcal{C}$  (see Definition 2.15) and let  $\mathcal{C}_{pt}$  be the maximal pointed subcategory of  $\mathcal{C}$ . Then

$$(69) \quad \mathcal{C}'_{ad} = \mathcal{C}_{pt} \quad \text{and} \quad \mathcal{C}'_{pt} = \mathcal{C}_{ad}.$$

For the proof of the following result see [26] and [9, Proposition 3.7].

**Proposition 4.29.** *The following conditions are equivalent for a pre-modular category  $\mathcal{C}$ :*

- (i)  $\mathcal{C}$  is modular;
- (ii)  $\mathcal{C}$  is non-degenerate, i.e.,  $\mathcal{C}' = \text{Vec}$ ;
- (iii)  $\mathcal{C}$  is factorizable, i.e., the functor  $G : \mathcal{C} \boxtimes \mathcal{C}^{rev} \rightarrow \mathcal{Z}(\mathcal{C})$  defined in (49) is an equivalence.

**Corollary 4.30.** *Let  $\mathcal{C}$  be a fusion category. Then its center  $\mathcal{Z}(\mathcal{C})$  is non-degenerate.*

*Proof.* It is proved in [14] that  $\mathcal{Z}(\mathcal{C})$  is factorizable, so the result follows from Proposition 4.29.  $\square$

The following Class Equation was proved in [11, Proposition 5.7]. It is very useful for classification of fusion categories, see Section 6.2 below.

**Theorem 4.31.** *Let  $\mathcal{A}$  be a spherical fusion category. Let  $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$  be the forgetful functor. Then*

$$(70) \quad \dim(\mathcal{A}) = \sum_{Z \in \mathcal{O}(\mathcal{Z}(\mathcal{A}))} [F(Z) : \mathbf{1}] d_Z,$$

and  $\frac{\dim(\mathcal{A})}{d_Z}$  is an algebraic integer for every  $Z \in \mathcal{O}(\mathcal{Z}(\mathcal{A}))$ .

*Proof.* Let  $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$  be the right adjoint of  $F$ . We have

$$(71) \quad FI(\mathbf{1}) \cong \bigoplus_{X \in \mathcal{O}(\mathcal{A})} X \otimes X^*$$

and, hence, the dimension of  $I(\mathbf{1})$  is equal to  $\sum_{X \in \mathcal{O}(\mathcal{A})} d_X^2 = \dim(\mathcal{A})$ . On the other hand,  $I(\mathbf{1}) \cong \bigoplus_{Z \in \mathcal{O}(\mathcal{Z}(\mathcal{A}))} [F(Z) : \mathbf{1}]Z$ . Taking the dimensions of both sides of the last equation we obtain (70). From Proposition 4.26 we know that  $\frac{\dim(\mathcal{Z}(\mathcal{A}))}{d_Z^2} = \left(\frac{\dim(\mathcal{A})}{d_Z}\right)^2$  is an algebraic integer.  $\square$

**Remark 4.32.** Theorem 4.31 says that  $\dim(\mathcal{A})$  can be written as a sum of algebraic integers that are also divisors of  $\dim(\mathcal{A})$  in the ring of algebraic integers. This is an analogue of the Class Equation in group theory. Indeed, when  $G$  is a finite group and  $\mathcal{A} = \text{Rep}(G)$  then simple subobjects of  $I(\mathbf{1})$  are in bijection with conjugacy classes of  $G$  and their dimensions are cardinalities of the corresponding conjugacy classes.

We have the following relation between the Frobenius-Perron and categorical dimensions, [11, Proposition 8.22].

**Proposition 4.33.** *For any spherical fusion category  $\mathcal{A}$  over  $\mathbb{C}$  the ratio  $\frac{\dim(\mathcal{A})}{\text{FPdim}(\mathcal{A})}$  is an algebraic integer  $\leq 1$ .*

*Proof.* Let  $\mathcal{C} = \mathcal{Z}(\mathcal{A})$ . By (21) it suffices to show that  $\frac{\dim(\mathcal{C})}{\text{FPdim}(\mathcal{C})}$  is an algebraic integer. Let  $S = \{s_{XY}\}$  denote the  $S$ -matrix of  $\mathcal{C}$ . The Frobenius-Perron dimension is a homomorphism from  $K(\mathcal{C})$  to  $\mathbb{C}$ , hence, it must be of the form (58). Thus, there exists a distinguished object  $X \in \mathcal{C}$  such that  $\text{FPdim}(Z) = \frac{s_{ZX}}{d_X}$  for all simple objects  $Z$  in  $\mathcal{C}$ . Therefore,

$$\text{FPdim}(\mathcal{C}) = \sum_Z \text{FPdim}(Z)^2 = \sum_Z \frac{s_{ZX}}{d_X} \frac{s_{Z^*X}}{d_X} = \frac{\dim(\mathcal{C})}{d_X^2}.$$

Thus,  $\frac{\dim(\mathcal{C})}{\text{FPdim}(\mathcal{C})} = d_X^2$ . The latter is an algebraic integer since  $d : K(\mathcal{C}) \rightarrow \mathbb{C}$  is a homomorphism.  $\square$

Now suppose that  $\mathcal{C}$  is a pseudo-unitary non-degenerate braided fusion category over  $\mathbb{C}$  (so that there is a spherical structure on  $\mathcal{C}$  such that  $d_X = \text{FPdim}(X)$ ). One can recognize pairs of centralizing simple objects of  $\mathcal{C}$  using the  $S$ -matrix.

**Proposition 4.34.** *Let  $X, Y$  be simple objects of  $\mathcal{C}$ . Then  $X$  centralizes  $Y$  if and only if  $s_{XY} = \text{FPdim}(X) \text{FPdim}(Y)$ .*

*Proof.* If  $X$  centralizes  $Y$  then  $c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}$  and

$$s_{XY} = d_X d_Y = \text{FPdim}(X) \text{FPdim}(Y).$$

Conversely, if  $s_{XY} = \text{FPdim}(X) \text{FPdim}(Y)$  then using formula (56) we obtain

$$\begin{aligned} \text{FPdim}(X) \text{FPdim}(Y) &= |s_{XY}| = |\theta_X^{-1} \theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^Z \theta_Z d_Z| \\ &\leq \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^Z \text{FPdim}(Z) = \text{FPdim}(X) \text{FPdim}(Y). \end{aligned}$$

The above inequality must be an equality, hence  $\frac{\theta_Z}{\theta_X \theta_Y} = 1$  for all simple objects  $Z$  contained in  $X \otimes Y$ . By (51) this means that  $c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}$ .  $\square$

**4.9. Equivariantization and de-equivariantization of braided fusion categories.** Let  $\mathcal{C}$  be a braided fusion category and let  $G$  be a group acting on  $\mathcal{C}$  by braided autoequivalences (i.e., each  $T_g$ ,  $g \in G$ , from (11) is a braided autoequivalence of  $\mathcal{C}$ ). Then the equivariantized fusion category  $\mathcal{C}^G$  inherits the braiding from  $\mathcal{C}$ . Note that  $\mathcal{C}^G$  contains a Tannakian subcategory  $\text{Rep}(G)$  such that  $\text{Rep}(G) \subset (\mathcal{C}^G)'$ .

Here we describe the converse construction, following [4, 26, 36]. Let  $\mathcal{D}$  be a braided fusion category containing a Tannakian subcategory  $\text{Rep}(G)$  such that

$$(72) \quad \text{Rep}(G) \subset \mathcal{D}'.$$

Let  $A$  be the algebra of functions on  $G$ . It is a commutative algebra in  $\text{Rep}(G)$  and, hence, in  $\mathcal{D}$ . The category  $\mathcal{D}_G$  of  $A$ -modules in  $\mathcal{D}$  has a canonical structure of a fusion category with the tensor product  $\otimes_A$ . Furthermore, condition (72) allows to define the braiding on  $\mathcal{D}_G$ . Thus,  $\mathcal{D}_G$  is a braided fusion category, called *de-equivariantization* of  $\mathcal{D}$ . There is a canonical action of  $G$  by braided autoequivalences of  $\mathcal{D}_G$  induced by the action of  $G$  on  $A$  by translations.

As the names suggest, the above two constructions are inverses of each other. Namely, there exist canonical braided equivalences of fusion categories:

$$(\mathcal{C}^G)_G \cong \mathcal{C} \quad \text{and} \quad (\mathcal{D}_G)^G \cong \mathcal{D}.$$

See [10, Section 4] for a complete treatment of equivariantization and de-equivariantization.

## 5. CHARACTERIZATION OF MORITA EQUIVALENCE

**5.1. Braided equivalences of centers.** The following theorem was proved in [12, Theorem 3.1]. It is a categorical counterpart of the well known fact in algebra that Morita equivalent rings have isomorphic centers.

**Theorem 5.1.** *Two fusion categories  $\mathcal{A}$  and  $\mathcal{B}$  are categorically Morita equivalent if and only if  $\mathcal{Z}(\mathcal{A})$  and  $\mathcal{Z}(\mathcal{B})$  are equivalent as braided fusion categories*

*Proof.* Let  $\mathcal{M}$  be an indecomposable left  $\mathcal{A}$ -module category such that  $\mathcal{B} = (\mathcal{A}_{\mathcal{M}}^*)^{\text{op}}$ . As in Section 3.2, we can view  $\mathcal{M}$  as an  $(\mathcal{A} \boxtimes \mathcal{B})$ -module category. It was observed in [37] (see also [28]) that the category of  $(\mathcal{A} \boxtimes \mathcal{B})$ -module endofunctors of  $\mathcal{M}$  can be identified, on the one hand, with functors of tensor multiplication by objects of  $\mathcal{Z}(\mathcal{A})$ , and on the other hand, with functors of tensor multiplication by objects of  $\mathcal{Z}(\mathcal{B})$ . Combined, these identifications yield a canonical equivalence of braided fusion categories

$$(73) \quad \mathcal{Z}(\mathcal{A}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{B}).$$

Conversely, let  $\mathcal{A}$  and  $\mathcal{B}$  be fusion categories such that there is a braided tensor equivalence  $a : \mathcal{Z}(\mathcal{B}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{A})$ . Let  $F : \mathcal{Z}(\mathcal{B}) \rightarrow \mathcal{B}$  be the forgetful functor and let  $I : \mathcal{B} \rightarrow \mathcal{Z}(\mathcal{B})$  be its right adjoint. Then  $I(\mathbf{1})$  is a commutative algebra in  $\mathcal{Z}(\mathcal{B})$  and  $L := a \circ I(\mathbf{1})$  is a commutative algebra in  $\mathcal{Z}(\mathcal{A})$ . We can also view  $L$  as an algebra in  $\mathcal{A}$ . Let

$$L = \oplus_i L_i$$

be the decomposition of  $L$  into the sum of indecomposable algebras in  $\mathcal{A}$  and let  $\mathcal{M}_i$  denote the category of right  $L_i$ -modules in  $\mathcal{A}$ . Then the equivalence class of  $\mathcal{M}_i$  does not depend on  $i$  and  $\mathcal{B} \cong \mathcal{A}_{\mathcal{M}_i}^*$ . See [12, Section 3] for details.  $\square$

Thus, the Morita equivalence class of a fusion category  $\mathcal{A}$  is completely determined by its center  $\mathcal{Z}(\mathcal{A})$ .

**Remark 5.2.** A more precise statement of Theorem 5.1 is given in [13, Theorem 1.1]. Namely, the 2-functor of taking the center gives a fully faithful embedding of the 2-category of Morita equivalences of fusion categories into the 2-category of braided fusion categories. In particular, for any fusion category  $\mathcal{A}$  the group  $\text{BrPic}(\mathcal{A})$  of Morita autoequivalences of  $\mathcal{A}$  is isomorphic to the group of braided autoequivalences of  $\mathcal{Z}(\mathcal{A})$  and there is an equivariant bijection between the set of Morita equivalences between  $\mathcal{A}$  and  $\mathcal{B}$  and braided equivalences (73). See [13, Section 5] for details.

**5.2. Recognizing centers of extensions.** Let  $G$  be a finite group and let  $\mathcal{A}$  be a  $G$ -extension of a fusion category  $\mathcal{B}$ :

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \mathcal{A}_e = \mathcal{B}.$$

The center of  $\mathcal{A}$  contains a Tannakian subcategory  $\mathcal{E} \cong \text{Rep}(G)$  whose objects are constructed as follows. For every representation  $\pi : G \rightarrow GL(V)$  of  $G$  consider the object  $Y_\pi$  in  $\mathcal{Z}(\mathcal{A})$ , where  $Y_\pi = V \otimes \mathbf{1}$  as an object of  $\mathcal{A}$  with the permutation isomorphism

$$\gamma_{Y_\pi} := \pi(g) \otimes \text{id}_X : X \otimes Y_\pi \xrightarrow{\sim} Y_\pi \otimes X, \quad X \in \mathcal{A}_g.$$

Here we identified  $X \otimes Y_\pi$  and  $Y_\pi \otimes X$  with  $V \otimes X$ .

The above property in fact characterizes  $G$ -extensions. The following statement is [12, Theorem 1.3].

**Theorem 5.3.** *Let  $G$  be a finite group. A fusion category  $\mathcal{A}$  is Morita equivalent to a  $G$ -extension of some fusion category if and only if  $\mathcal{Z}(\mathcal{A})$  contains a Tannakian subcategory  $\mathcal{E} = \text{Rep}(G)$ .*

**Remark 5.4.** In the situation of Theorem 5.3 consider the de-equivariantization  $\mathcal{E}'_G$ , see Section 4.9. Then  $\mathcal{A}$  is Morita equivalent to a  $G$ -graded fusion category  $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$  with  $\mathcal{Z}(\mathcal{B}_e) \cong \mathcal{E}'_G$ . In particular,

$$(74) \quad \text{FPdim}(\mathcal{B}_e) = \frac{\text{FPdim}(\mathcal{A})}{\text{FPdim}(\mathcal{E})}.$$

**Corollary 5.5.** *A fusion category  $\mathcal{A}$  is group-theoretical if and only if  $\mathcal{Z}(\mathcal{A})$  contains a Tannakian subcategory  $\mathcal{E}$  such that  $\text{FPdim}(\mathcal{E}) = \text{FPdim}(\mathcal{A})$ .*

*Proof.* From (74) we see that  $\mathcal{A}$  is categorically Morita equivalent to a  $G$ -extension whose trivial component is  $\text{Vec}$ . Any category with the latter property is pointed and is equivalent to  $\text{Vec}_G^\omega$  for some 3-cocycle  $\omega$ .  $\square$

## 6. CLASSIFICATION RESULTS FOR FUSION CATEGORIES OF INTEGRAL DIMENSIONS

**6.1. Fusion categories of integral dimension.** The following result is proved in [11, Proposition 8.24].

**Proposition 6.1.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A})$  is an integer. Then  $\mathcal{A}$  is pseudo-unitary.*

*Proof.* It is shown in Proposition 4.33 that the ratio  $\frac{\dim(\mathcal{A})}{\text{FPdim}(\mathcal{A})}$  is an algebraic integer  $\leq 1$ . Let  $D := \dim(\mathcal{A})$ , let  $D_1 = D, D_2, \dots, D_N$  be algebraic conjugates of  $D$ , and let  $g_1, \dots, g_N$  be the elements of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  such that  $D_i = g_i(D)$ . Applying Proposition 4.33 to the category  $g_i(\mathcal{A})$  we see that  $\frac{\dim(g_i(\mathcal{A}))}{\text{FPdim}(\mathcal{A})}$  is an algebraic integer  $\leq 1$ . Therefore,

$$\prod_{i=1}^N \frac{\dim(g_i(\mathcal{A}))}{\text{FPdim}(\mathcal{A})}$$

is an algebraic integer  $\leq 1$ . But this product is a rational number. It must be equal to 1 and so all factors are equal to 1. Thus,  $\dim(\mathcal{A}) = \text{FPdim}(\mathcal{A})$ , as desired.  $\square$

**Corollary 6.2.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A})$  is an integer. Then  $\mathcal{A}$  admits a unique spherical structure  $a_X : X \xrightarrow{\sim} X^{**}$  with respect to which  $d_X = \text{FPdim}(X)$  for every simple object  $X$ .*

*Proof.* This follows from Propositions 2.34 and 6.1.  $\square$

**Proposition 6.3.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A})$  is an integer.*

- (i) *For any  $X \in \mathcal{O}(\mathcal{A})$  we have  $\text{FPdim}(X) = \sqrt{n_X}$  for some integer  $n_X$ .*
- (ii) *The map  $\deg : \mathcal{O}(\mathcal{A}) \rightarrow \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$  that takes  $X \in \mathcal{O}(\mathcal{A})$  to the image of  $\text{FPdim}(X)$  in  $\mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$  is a grading of  $\mathcal{A}$ .*

*Proof.* Note that if  $Y$  is an object of  $\mathcal{A}$  such that  $\text{FPdim}(Y) \in \mathbb{Z}$  then  $\text{FPdim}(Y_0) \in \mathbb{Z}$  for any subobject  $Y_0$  of  $Y$ . Take  $Y = \bigoplus_{X \in \mathcal{O}(\mathcal{A})} X \otimes X^*$ . We have  $\text{FPdim}(Y) = \text{FPdim}(\mathcal{A}) \in \mathbb{Z}$ . But  $X \otimes X^*$  is a subobject of  $Y$  for every  $X \in \mathcal{O}(\mathcal{C})$ . Hence,  $\text{FPdim}(X)^2 = \text{FPdim}(X \otimes X^*) \in \mathbb{Z}$ . This proves the first part. The second part is clear (cf. [18, Theorem 3.10]).  $\square$

**Corollary 6.4.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A})$  is odd. Then  $\text{FPdim}(X) \in \mathbb{Z}$  for any  $X \in \mathcal{O}(\mathcal{C})$ .*

*Proof.* If  $\mathcal{A}$  contains objects of irrational dimension then by Proposition 6.3 it has a non-trivial grading by a 2-group and so  $\text{FPdim}(\mathcal{A})$  is even.  $\square$

**Remark 6.5.** Let  $\mathcal{A}_{ad}$  be the adjoint subcategory of  $\mathcal{A}$ , see Definition 2.15. The proof of Proposition 6.3 shows that if  $\text{FPdim}(\mathcal{A}) \in \mathbb{Z}$  then  $\mathcal{A}_{ad}$  is integral.

**6.2. Fusion categories of prime power Frobenius-Perron dimension.** Let  $p$  be a prime number.

The following result is proved in [11, Theorem 8.28]. It is a generalization of the result of Masuoka [23] in the theory of semisimple Hopf algebras.

**Proposition 6.6.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A}) = p^n$  for  $n \geq 1$ . Then  $\mathcal{A}$  has a faithful grading by  $\mathbb{Z}/p\mathbb{Z}$ .*

*Proof.* By Corollary 6.2,  $\mathcal{A}$  has a spherical structure such that  $d_X = \text{FPdim}(X)$  for all objects  $X$  of  $\mathcal{A}$ . In particular,  $\mathcal{Z}(\mathcal{A})$  is a modular category. Let  $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$  and  $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$  be the forgetful functor and its right adjoint.

Let  $Z \in \mathcal{O}(\mathcal{Z}(\mathcal{A}))$  be a simple subobject of  $I(\mathbf{1})$ . Since  $F(Z) \in \mathcal{A}_{ad}$  we conclude by Remark 6.5 that  $\text{FPdim}(Z)$  is an integer. By Proposition 4.26,  $\text{FPdim}(Z)$  is a power of  $p$ . Therefore, the right hand side of the Class Equation (70) must have at least  $p$  summands equal to 1. These summands correspond to distinct invertible simple subobjects of  $I(\mathbf{1})$ . Hence, the Abelian group  $G$  of such objects is non-trivial.

Elements of  $G$  are in bijection with tensor automorphisms of  $\text{id}_{\mathcal{A}}$ . Consequently,  $\mathcal{A}$  has a faithful grading by  $\widehat{G}$ . Since  $G$  is a  $p$ -group, it has a quotient isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  that provides a desired grading.  $\square$

**Corollary 6.7.** *A fusion category of prime power Frobenius-Perron dimension is nilpotent.*

**Corollary 6.8.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A}) = p$ . Then  $\mathcal{A}$  is equivalent to  $\text{Vec}_{\mathbb{Z}/p\mathbb{Z}}^{\omega}$  for some  $\omega \in Z^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{k}^{\times})$ .*

**Remark 6.9.** Corollary 6.8 is a generalization of the classical result of Kac [20] and Zhu [43] saying that a Hopf algebra of prime dimension is isomorphic to the group algebra of  $\mathbb{Z}/p\mathbb{Z}$ .

Recall from Definition 3.18 that a fusion category  $\mathcal{A}$  is group-theoretical if it is categorically Morita equivalent to a pointed fusion category.

Let  $\mathcal{C}$  be a braided fusion category. For any fusion subcategory  $\mathcal{L} \subset \mathcal{C}$  let  $\mathcal{L}^{co}$  denote fusion subcategory of  $\mathcal{C}$  generated by simple objects  $X \in \mathcal{O}(\mathcal{C})$  such that  $X \otimes X^* \in \mathcal{L}$ .

**Remark 6.10.** The superscript  $co$  stands for “commutator”. The reason is that for  $\mathcal{C} = \text{Rep}(G)$  and  $\mathcal{L} = \text{Rep}(G/N)$ , where  $N$  is a normal subgroup of  $G$ , one has  $\mathcal{L}^{co} = \text{Rep}(G/[G, N])$ , where  $[G, N]$  is the commutator subgroup.

Theorem 6.12 is proved in [9]. Below we sketch its proof. The reader is referred to [9] for full details.

**Lemma 6.11.** *Let  $\mathcal{C}$  be a nilpotent braided fusion category. There exists a symmetric subcategory  $\mathcal{K} \subset \mathcal{C}$  such that  $(\mathcal{K}')_{ad} \subset \mathcal{K}$ .*

*Proof.* Let  $\mathcal{K}$  be a symmetric subcategory of  $\mathcal{C}$ . If the condition  $(\mathcal{K}')_{ad} \subset \mathcal{K}$  is not satisfied then fusion subcategory  $\mathcal{E} \subset \mathcal{C}$  generated by  $\mathcal{K}$  and  $\mathcal{K}^{co} \cap (\mathcal{K}^{co})'$  is symmetric and  $\mathcal{K} \subsetneq \mathcal{E}$ . Thus, any maximal symmetric subcategory of  $\mathcal{C}$  satisfies the condition of the Lemma.  $\square$

**Theorem 6.12.** *Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(X) \in \mathbb{Z}$  for all  $X \in \mathcal{O}(\mathcal{A})$  and  $\text{FPdim}(\mathcal{A}) = p^n$  for some prime  $p$ . Then  $\mathcal{A}$  is group-theoretical.*

*Proof.* By Corollary 5.5 it suffices to show that non-degenerate braided fusion category  $\mathcal{C} := \mathcal{Z}(\mathcal{Z}(\mathcal{A})) \cong \mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{A})^{\text{rev}}$  contains a Tannakian subcategory  $\mathcal{E}$  such that  $\text{FPdim}(\mathcal{E}) = \text{FPdim}(\mathcal{A})^2$ .

This is achieved as follows. By Lemma 6.11,  $\mathcal{Z}(\mathcal{A})$  contains a symmetric subcategory  $\mathcal{K}$  such that  $(\mathcal{K}')_{ad} \subset \mathcal{K}$ . This means that there is a faithful grading

$$\mathcal{K}' = \bigoplus_{g \in G} \mathcal{K}'_g, \quad \text{with} \quad \mathcal{K}'_e = \mathcal{K}.$$

We view  $(\mathcal{K}')^{\text{rev}}$  as a fusion subcategory of  $\mathcal{Z}(\mathcal{A})^{\text{rev}}$  and set

$$\mathcal{E} := \bigoplus_{g \in G} \mathcal{K}'_g \boxtimes (\mathcal{K}'_g)^{\text{rev}} \subset \mathcal{C}.$$

Then  $\mathcal{E}$  is a symmetric subcategory such that  $\mathcal{E}' = \mathcal{E}$  (so that  $\text{FPdim}(\mathcal{E}) = \text{FPdim}(\mathcal{A})^2$  by (66)). In the case when  $p$  is odd this subcategory  $\mathcal{E}$  is automatically Tannakian. When  $p = 2$  one can show that existence of such  $\mathcal{E}$  implies existence of a Tannakian subcategory of the same dimension.  $\square$



**6.3. Symmetric subcategories of integral braided fusion categories.** We have seen in Theorem 5.3 that non-trivial Tannakian subcategories of the center of a fusion category  $\mathcal{A}$  are quite helpful in the study of the categorical Morita equivalence class of  $\mathcal{A}$ . In this Section we recall results of [12, Section 7] that establish existence of Tannakian subcategories of integral modular categories under certain assumptions on dimensions of their objects.

By Corollary 6.2 there is a canonical spherical structure on  $\mathcal{A}$  such that  $d_X = \text{FPdim}(X)$  for all objects  $X$  in  $\mathcal{A}$  and  $\dim(\mathcal{C}) = \text{FPdim}(\mathcal{C})$ . So we will simply talk about *dimensions* of objects and fusion categories.

Let  $\mathcal{C}$  be a non-degenerate integral braided fusion category with braiding

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X.$$

By Proposition 4.29 the category  $\mathcal{C}$  is modular. Let  $S = \{s_{XY}\}$  denote the  $S$ -matrix of  $\mathcal{C}$ . By (56) each entry  $s_{XY}$  is a sum of  $d_X d_Y$  roots of unity, so it can be viewed as a complex number. We have

$$(75) \quad |s_{XY}| \leq d_X d_Y,$$

where  $|z|$  denotes the absolute value of  $z \in \mathbb{C}$ .

**Lemma 6.13.** *Let  $X, Y$  be simple objects of  $\mathcal{C}$ . The following conditions are equivalent:*

- (i)  $|s_{XY}| = d_X d_Y$ ,
- (ii)  $c_{Y,X} \circ c_{X,Y}$  equals  $\text{id}_{X \otimes Y}$  times a scalar,
- (iii)  $X$  centralizes  $Y \otimes Y^*$ ,
- (iv)  $Y$  centralizes  $X \otimes X^*$ ,

*Proof.* See [18, Lemma 6.5] and [10, Proposition 3.32].  $\square$

**Definition 6.14.** When equivalent conditions of Lemma 6.13 are satisfied, we say that  $X$  and  $Y$  *projectively centralize* each other.

**Lemma 6.15.** *Let  $X$  and  $Y$  be two simple objects of  $\mathcal{C}$  such that  $d_X$  and  $d_Y$  are relatively prime. Then one of two possibilities hold:*

- (i)  $X, Y$  projectively centralize each other, or
- (ii)  $s_{XY} = 0$ .

*Proof.* By (58),  $\frac{s_{XY}}{d_X}$  and  $\frac{s_{XY}}{d_Y}$  are algebraic integers. Since  $d_X$  and  $d_Y$  are relatively prime,  $\alpha := \frac{s_{XY}}{d_X d_Y}$  is also an algebraic integer. Indeed, if  $a, b \in \mathbb{Z}$  are such that  $ad_X + bd_Y = 1$  then

$$\frac{s_{XY}}{d_X d_Y} = a \frac{s_{XY}}{d_Y} + b \frac{s_{XY}}{d_X}.$$

Let  $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n$  be algebraic conjugates of  $\alpha$ . Then the norm  $\alpha_1 \alpha_2 \cdots \alpha_n$  is an integer. Since it is  $\leq 1$  in absolute value it must be either  $\pm 1$  or  $0$ . In the former case  $|\alpha_i| = 1$  for every  $i$  and so  $|\alpha| = 1$ , i.e.,  $X, Y$  projectively centralize each other. In the latter case  $\alpha = 0$ , i.e.,  $s_{XY} = 0$ .  $\square$

**Corollary 6.16.** *Suppose  $\mathcal{C}$  contains a simple object  $X$  with dimension  $d_X = p^r$ , where  $p$  is a prime and  $r > 0$ . Then  $\mathcal{C}$  contains a nontrivial symmetric subcategory.*

*Proof.* We first show that  $\mathcal{C}$  contains a nontrivial proper subcategory. Let us assume that it does not. Take any simple  $Y \neq \mathbf{1}$  with  $d_Y$  coprime to  $d_X$  (such a  $Y$  must exist since  $p$  divides  $\dim(\mathcal{C})$  by Proposition 4.26). We claim that  $s_{XY} = 0$ . Indeed, otherwise  $X$  and  $Y$  projectively centralize each other by Lemma 6.15, so

the centralizer of the category generated by  $Y \otimes Y^*$  is nontrivial, and we get a nontrivial proper subcategory, a contradiction.

Now let us use the orthogonality of columns  $(s_{XY})$  and  $(d_Y)$  of the  $S$ -matrix:

$$\sum_{Y \in \mathcal{O}(\mathcal{C})} \frac{s_{XY}}{d_X} d_Y = 0.$$

It follows that all the nonzero summands in this sum, except the one for  $Y = \mathbf{1}$ , come from objects  $Y$  of dimension divisible by  $p$ . Therefore, all the summands in this sum except for the one for  $Y = \mathbf{1}$  (which equals 1) are divisible by  $p$ . This is a contradiction.

Now we prove the corollary by induction in  $\dim(\mathcal{C})$ . Let  $\mathcal{D}$  be a nontrivial proper subcategory of  $\mathcal{C}$ . If  $\mathcal{D}$  is degenerate, then  $\mathcal{D} \cap \mathcal{D}'$  is a nontrivial proper symmetric subcategory of  $\mathcal{C}$ , so we are done. Otherwise,  $\mathcal{D}$  is non-degenerate, and by (68) we have  $\mathcal{C} = \mathcal{D} \boxtimes \mathcal{D}'$ . Thus  $X = X_1 \otimes X_2$ , where  $X_1 \in \mathcal{D}$ ,  $X_2 \in \mathcal{D}'$  are simple. Since the dimension of  $X_1$  or  $X_2$  is a positive power of  $p$ , we get the desired statement from the induction assumption applied to  $\mathcal{D}$  or  $\mathcal{D}'$  (which are non-degenerate braided fusion categories of smaller dimension).  $\square$

**Remark 6.17.** Corollary 6.16 generalizes Burnside's theorem that a finite group  $G$  with a conjugacy class of prime power size can not be simple.

**6.4. Solvability of fusion categories of Frobenius-Perron dimension  $p^a q^b$ .** Recall from Definition 3.23 that a fusion category  $\mathcal{A}$  is solvable if it is categorically Morita equivalent to a cyclically nilpotent fusion category.

**Proposition 6.18.** *A fusion category  $\mathcal{A}$  is solvable if and only if there is a sequence of fusion categories*

$$\mathcal{A}_0 = \text{Vec}, \mathcal{A}_1, \dots, \mathcal{A}_n = \mathcal{A}$$

*and a sequence of cyclic groups of prime order such that  $\mathcal{A}_i$  is obtained from  $\mathcal{A}_{i-1}$  either by a  $G_i$ -equivariantization or as a  $G_i$ -extension.*

*Proof.* See [12, Proposition 4.4].  $\square$

Let  $p$  and  $q$  be prime numbers.

**Theorem 6.19.** *Let  $\mathcal{C}$  be an integral non-degenerate braided fusion category of Frobenius-Perron dimension  $p^a q^b$ . If  $\mathcal{C}$  is not pointed, it contains a Tannakian subcategory  $\text{Rep}(G)$ , where  $G$  is a cyclic group of prime order.*

*Proof.* First let us show that  $\mathcal{C}$  contains an invertible object. Assume the contrary. By Proposition 4.26 the dimension of every  $X \in \mathcal{O}(\mathcal{C})$  divides  $p^a q^b$ . There must be a simple object in  $\mathcal{C}$  whose dimension is a prime power, since otherwise the dimension of every non-identity simple object is divisible by  $pq$  and

$$\text{FPdim}(\mathcal{C}) = 1 \pmod{pq},$$

a contradiction. By Proposition 6.16  $\mathcal{C}$  contains a non-trivial symmetric subcategory  $\mathcal{E}$ . This category  $\mathcal{E}$  is not equivalent to  $\text{sVec}$  since  $\mathcal{C}_{pt} = \text{Vec}$  by assumption. By Remark 4.7  $\mathcal{E}$  contains a non-trivial Tannakian subcategory  $\text{Rep}(G)$ . The group  $G$  is solvable by the classical Burnside's theorem in group theory, hence,  $\text{Rep}(G)$  must contain invertible objects.

Let  $\mathcal{C}_{pt}$  be the maximal pointed subcategory of  $\mathcal{C}$ . We claim that  $\mathcal{C}_{pt}$  cannot be non-degenerate. Indeed, otherwise  $\mathcal{C} \cong \mathcal{C}_{pt} \boxtimes \mathcal{C}_1$ , where  $\mathcal{C}_1 = \mathcal{C}'_{pt}$  by (68). But then the above argument  $\mathcal{C}_1$  contains non-trivial invertible objects which is absurd.

Consider the symmetric fusion category  $\mathcal{E} = \mathcal{C}_{pt} \cap \mathcal{C}'_{pt}$ . Let  $n = \text{FPdim}(\mathcal{E})$ . If  $n > 2$  then  $\mathcal{E}$  contains a non-trivial Tannakian subcategory. It remains to consider the case when  $n = 2$ , i.e., when  $\mathcal{E} = \mathbf{sVec}$  (this can only happen if one of the primes  $p, q$  is equal to 2). This situation is treated in [12, Propositions 7.4 and 8.3].  $\square$

**Corollary 6.20.** *A fusion category of Frobenius-Perron dimension  $p^a q^b$  is solvable.*

*Proof.* Let  $\mathcal{A}$  be a fusion category such that  $\text{FPdim}(\mathcal{A}) = p^a q^b$ . We may assume that  $\mathcal{A}$  is integral. Indeed, otherwise  $\mathcal{A}$  is an extension of an integral fusion category by Remark 6.5 and the result follows by induction on  $\text{FPdim}(\mathcal{A})$  using Lemma 3.25.

The category  $\mathcal{Z}(\mathcal{A})$  satisfies the hypothesis of Theorem 6.19 and, hence, contains a Tannakian subcategory  $\text{Rep}(G)$ , where  $G$  is a cyclic group of prime order. By Theorem 5.3  $\mathcal{A}$  is categorically Morita equivalent to a  $G$ -extension of some fusion category  $\mathcal{B}$ . Since  $\text{FPdim}(\mathcal{B}) \leq \text{FPdim}(\mathcal{A})$  the result follows by induction.  $\square$

**6.5. Other results and open problem.** Using orthogonality of columns of the  $S$ -matrix of a modular category one can prove several other classification results. In particular, it was shown in [12] that integral fusion categories of dimension  $pqr$ , where  $p, q, r$  are primes are group-theoretical. It was also shown there that fusion categories of dimension 60 are weakly group-theoretical.

In a different direction, it was shown in [31] that integral braided fusion category  $\mathcal{A}$  such that every simple object of  $\mathcal{A}$  has Frobenius-Perron dimension at most 2 is solvable.

The following natural question was asked in [12]. The answer is unknown to the author.

**Question 6.21.** Is every integral fusion category weakly group-theoretical?

Perhaps the arithmetic properties of  $S$ -matrices discussed in Section 4.7 can be used in order to answer this question.

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